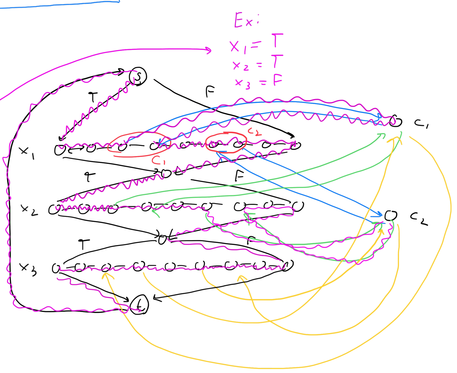
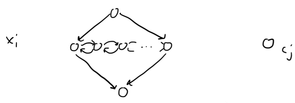
Theorem 7.46: 3SAT reduces to Hamiltonian Cycle in a directed graph (so HC is NP-Hard)

Proof: We begin with a 3SAT formula $\psi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \dots \wedge (a_k \vee b_k \vee c_k)$. We let each a, b, c denote a literal x_i or $\overline{x_i}$. Let $x_1 \dots x_l$ be l variables of ψ.

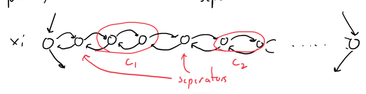
Ex: $\psi = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3)$.



We will convert ψ into a graph G that has a HC iff ψ is satisfiable. We convert each variable x_i into a "diamond-like" gadget. Each clause of ψ is converted to a single vertex c_j ($1 \le j \le k$).



Each diamond has a horizontal row of $3k+3$ vertices. Each c_j is represented via a pair, with an extra seperator vertex.



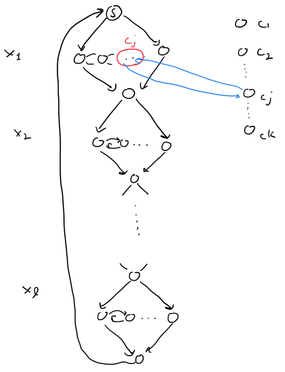
If x_i appears in clause c_j of ψ, then 2 edges are added to G from the j-th pair of the i-th diamond to the j-th clause.



If $\overline{x_i}$ appears in clause c_j, we add 2 edges in the reverse direction.

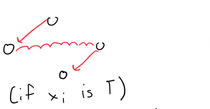


We end up with G as such:



(sufficiency)

$\Rightarrow$ Suppose ψ is satisfiable. We can demonstrate a HC in G. Beginning from s, we go through each diamond and end at t. There are 2 ways to go through a diamond.



We can easily update the path to include the clause vertices via a detour. That is, if $x_i = T$, we take the 2 edges to include c_j via the c_j pair in the i-th diamond.

(necessity)

$\Leftarrow$ Suppose G has a HC P. We will show it satisfies ψ. If P goes left-to-right in diamond i, then we assign True to the corresponding $x_i \in \psi$. If P goes right-to-left in diamond i, then we assign x_i = False in ψ. Since each clause appears in P, we can determine which literals in the corresponding clause are True. $\square$

Corollary: HC in a directed graph is NPC.

Proof: Let G be a directed graph. We are given a cycle P. We can easily check whether P is a HC in $O(n)$ by iterating on all vertices in P and comparing $|P|$ to $|V|$. This shows that HC is in NP. We previously proved it is NP-Hard. Thus, it is NPC. $\square$