$$x_F(t) = X_c[0] + \sum_{k=1}^{\infty} [X_c[k] \cos(2\pi(kf_F)t) + X_s[k] \sin(2\pi(kf_F)t)] \quad \textbf{(4.36)}$$

$$X_c[k] = 2\,\mathrm{Re}(X[k]) = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t) \cos(2\pi(kf_F)t)\, dt \quad \textbf{(4.37)}$$

$$X_s[k] = -2\,\mathrm{Im}(X[k]) = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t) \sin(2\pi(kf_F)t)\, dt. \quad \textbf{(4.38)}$$

$$X[0] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t)\, dt, \quad \textbf{(4.33)}$$

$$\begin{aligned} X_c[0] &= X[0] \\ X_s[0] &= 0 \\ X_c[k] &= X[k] + X^*[k] \\ X_s[k] &= j\,(X[k] - X^*[k]) \qquad k = 1, 2, 3, \ldots \end{aligned} \quad \textbf{(4.39)}$$

$$\begin{aligned} X[0] &= X_c[0] \\ X[k] &= \frac{X_c[k] - jX_s[k]}{2} \qquad k = 1, 2, 3, \ldots \\ X[-k] = X^*[k] &= \frac{X_c[k] + jX_s[k]}{2} \end{aligned} \quad \textbf{(4.40)}$$

Linearity	$\alpha x(t) + \beta y(t) \overset{\mathcal{FS}}{\longleftrightarrow} \alpha X[k] + \beta Y[k]$
Time shifting	$x(t - t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-j2\pi(kf_0)t_0} X[k]$
Time reversal	$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} X[-k]$
Time scaling	If $z(t) = x(at)$, $a > 0$, then
	a. On the period $T_F = T_0/a$:
	$Z[k] = X[k]$
	and $z(t) = x(at) = \sum_{k=-\infty}^{\infty} Z[k] e^{j2\pi(akf_0)t}$

and $z(t) = x(at) = \sum_{k=-\infty}^{\infty} Z[k] e^{j2\pi(akf_0)t}$

b. On the period $T_F = T_0$:

$$Z[k] = \begin{cases} X\left[\frac{k}{a}\right] & \frac{k}{a} \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and} \quad z(t) = x(at) = \sum_{k=-\infty}^{\infty} Z[k]e^{j2\pi(kf_0)t}.$$

Change of representation period	On the period $T_F = mT_0$, where m is a positive integer, $$X_m[k] = \begin{cases} X\left[\frac{k}{m}\right] & \frac{k}{m} \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$
Time differentiation	$$\frac{d}{dt}(x(t)) \xleftrightarrow{\mathcal{FS}} j2\pi(kf_0)X[k]$$
Time integration	$$\int_{-\infty}^{t} x(\lambda)\, d\lambda \xleftrightarrow{\mathcal{FS}} \frac{X[k]}{j2\pi(kf_0)} \qquad \text{if } X[0] = 0$$
Multiplication–convolution duality	$$x(t)y(t) \xleftrightarrow{\mathcal{FS}} \sum_{q=-\infty}^{\infty} Y[q]X[k-q] = X[k] * Y[k]$$ $$x(t) \circledast y(t) = \int_{T_0} x(\tau)y(t-\tau)\, d\tau \xleftrightarrow{\mathcal{FS}} T_0X[k]Y[k]$$
Conjugation	$$x^*(t) \xleftrightarrow{\mathcal{FS}} X^*[-k]$$
Parseval's theorem	$$\frac{1}{T_0}\int_{T_0} \lvert x(t)\rvert^2\, dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$$

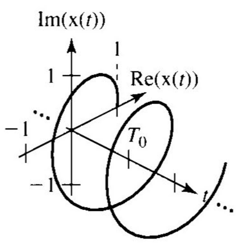
E.1 FOURIER SERIES

Continuous-Time Fourier Series (CTFS)

The following Fourier pairs are for a periodic CT function represented over the period T_F.

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{j2\pi(kf_F)t} \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k] = \frac{1}{T_F} \int_{T_F} \mathrm{x}(t) e^{-j2\pi(kf_F)t}\, dt$$

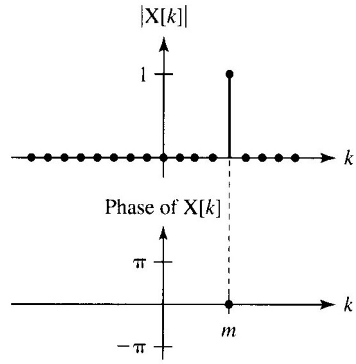
$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{j(k\omega_F)t} \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k] = \frac{1}{T_F} \int_{T_F} \mathrm{x}(t) e^{-j(k\omega_F)t}\, dt$$

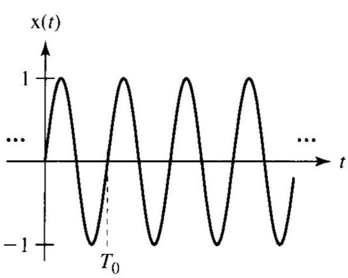


$$T_F = mT_0$$

$$e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{FS}} \delta[k-m]$$

$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{FS}} \delta[k-m]$$



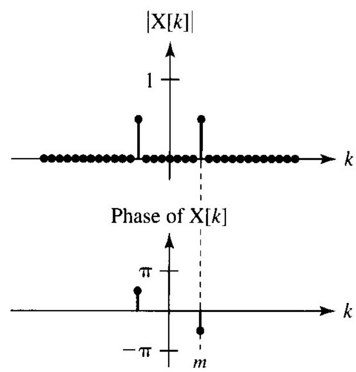


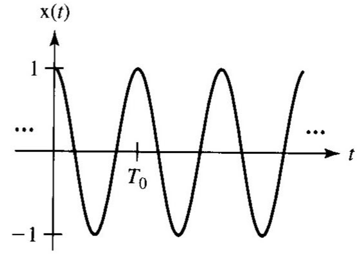
$$T_F = mT_0$$

$$\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{FS}} \frac{j}{2}(\delta[k+m] - \delta[k-m])$$

$$\sin(\omega_0 t) \xleftrightarrow{\mathcal{FS}} \frac{j}{2}(\delta[k+m] - \delta[k-m])$$

(m is an integer)



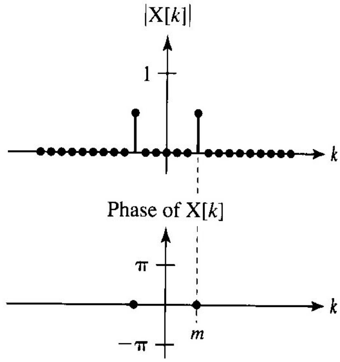


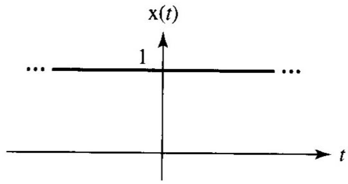
$$T_F = mT_0$$

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{FS}} \frac{1}{2}(\delta[k-m] + \delta[k+m])$$

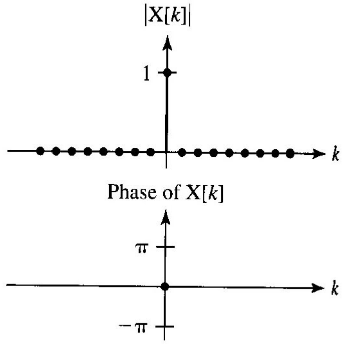
$$\cos(\omega_0 t) \xleftrightarrow{\mathcal{FS}} \frac{1}{2}(\delta[k-m] + \delta[k+m])$$

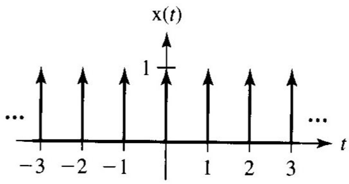
(m is an integer)





$$1 \xleftrightarrow{\mathcal{FS}} \delta[k]$$

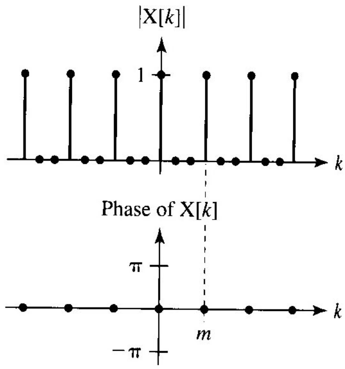


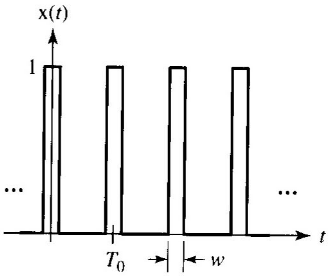


$$T_F = mT_0 = m$$

$$\mathrm{comb}(t) \xleftrightarrow{\mathcal{FS}} \mathrm{comb}_m[k]$$

(m is an integer)

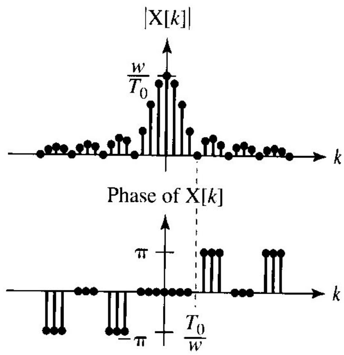


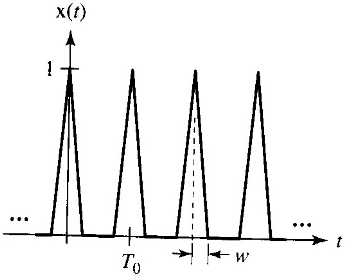


$$T_F = T_0$$

$$\mathrm{rect}\left(\frac{t}{w}\right) * \frac{1}{T_0}\,\mathrm{comb}\left(\frac{t}{T_0}\right) \xleftrightarrow{\mathcal{FS}} \frac{w}{T_0}\,\mathrm{sinc}\left(\frac{w}{T_0}k\right)$$

$$\mathrm{rect}\left(\frac{t}{w}\right) * f_0\,\mathrm{comb}(f_0 t) \xleftrightarrow{\mathcal{FS}} w f_0\,\mathrm{sinc}(w k f_0)$$

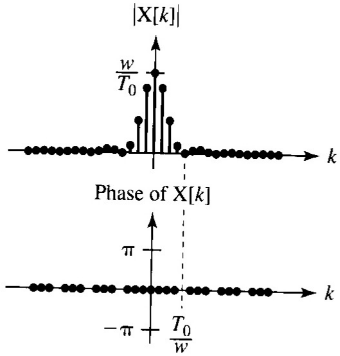


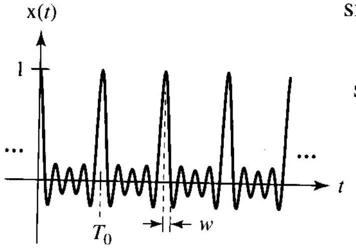


$$T_F = T_0$$

$$\mathrm{tri}\left(\frac{t}{w}\right) * \frac{1}{T_0}\,\mathrm{comb}\left(\frac{t}{T_0}\right) \xleftrightarrow{\mathcal{FS}} \frac{w}{T_0}\,\mathrm{sinc}^2\left(\frac{w}{T_0}k\right)$$

$$\mathrm{tri}\left(\frac{t}{w}\right) * f_0\,\mathrm{comb}(f_0 t) \xleftrightarrow{\mathcal{FS}} w f_0\,\mathrm{sinc}^2(w k f_0)$$





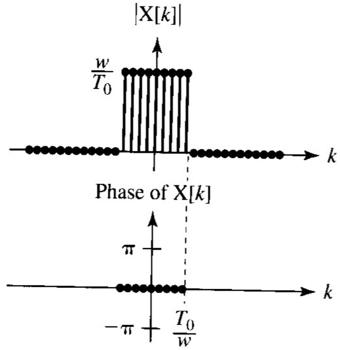
$$T_F = T_0$$

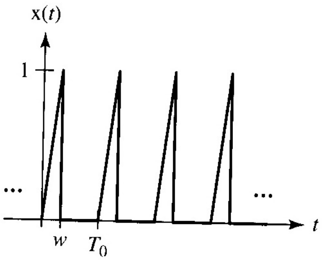
$$\mathrm{sinc}\left(\frac{t}{w}\right) * \frac{1}{T_0}\,\mathrm{comb}\left(\frac{t}{T_0}\right) \xleftrightarrow{\mathcal{FS}} \frac{w}{T_0}\,\mathrm{rect}\left(\frac{w}{T_0}k\right)$$

$$\mathrm{sinc}\left(\frac{t}{w}\right) * f_0\,\mathrm{comb}(f_0 t) \xleftrightarrow{\mathcal{FS}} w f_0\,\mathrm{rect}(w k f_0)$$

$$w f_0 (2M+1)\,\mathrm{drcl}(f_0 t, 2M+1) \xleftrightarrow{\mathcal{FS}} w f_0\,\mathrm{rect}(w k f_0)$$

$$\left(M \text{ is the greatest integer in } \frac{T_0}{2w}\right)$$





$$T_F = T_0$$

$$\frac{t}{w}(\mathrm{u}(t) - \mathrm{u}(t-w)) * \frac{1}{T_0}\,\mathrm{comb}\left(\frac{t}{T_0}\right) \xleftrightarrow{\mathcal{FS}}$$

$$\frac{1}{w T_0}\,\frac{(j(2\pi k w/T_0) + 1)e^{-j(2\pi k w/T_0)} - 1}{(2\pi k/T_0)^2}$$