MATHEMATICAL METHODS

1. Sums of Powers of Integers

2. Logarithms

3. Permutations, Combinations, Factorials

4. Fibonacci Numbers

5. Catalan Numbers
THEOREM A.1

\[ 1 + 2 + \cdots + n = \frac{n(n + 1)}{2}. \]
\[ 1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}. \]

PROOF:

\[
\begin{align*}
1 + 2 + 3 + \cdots + n - 1 + n &= S \\
n + n - 1 + n - 2 + \cdots + 2 + 1 &= S \\
n + 1 + n + 1 + n + 1 + \cdots + n + 1 + n + 1 &= 2S
\end{align*}
\]

There are \( n \) columns on the left; hence \( n(n + 1) = 2S \) and the first formula follows.
### Theorem A.2

1 + 2 + 4 + \cdots + 2^{m-1} = 2^m - 1.

1 \times 1 + 2 \times 2 + 3 \times 4 + \cdots + m \times 2^{m-1} = (m - 1) \times 2^m + 1.

In summation notation these equations are

\[
\sum_{k=0}^{m-1} 2^k = 2^m - 1.
\]

\[
\sum_{k=1}^{m} k \times 2^{k-1} = (m - 1) \times 2^m + 1.
\]

---

### Theorem A.3

If \(|x| < 1\) then

\[
\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} \quad \text{and} \quad \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1 - x)^2}.
\]
Logarithms are defined in terms of a real number \( a > 1 \), which is called the base of the logarithms. For any number \( x > 0 \), we define \( \log_a x = y \), where \( y \) is the real number such that \( a^y = x \). The logarithm of a negative number, and the logarithm of 0, are not defined.

\[
\begin{align*}
\log_a 1 &= 0, \\
\log_a a &= 1, \\
\log_a x &< 0 \quad \text{for all } x \text{ such that } 0 < x < 1. \\
0 &< \log_a x < 1 \quad \text{for all } x \text{ such that } 1 < x < a. \\
\log_a x &> 1 \quad \text{for all } x \text{ such that } a < x.
\end{align*}
\]

\[
\begin{align*}
\log_a (xy) &= (\log_a x) + (\log_a y) \\
\log_a (x/y) &= (\log_a x) - (\log_a y) \\
\log_a x^z &= z \log_a x \\
\log_a a^z &= z \\
a^{\log_a x} &= x,
\end{align*}
\]

where \( x, y, \) and \( z \) are real numbers, \( x > 0 \) and \( y > 0 \).

As \( x \) grows large, 
\[
\log x \text{ grows more slowly than } x^c, \text{ for any } c > 0.
\]
The Base for Logarithms

- Base 10 gives *common logarithms*; often used for hand computation and for expressing very large or small numbers.

- Base \( e = 2.718281828459 \ldots \) gives *natural logarithms*; it appears often in the mathematical analysis of algorithms; denoted \( \ln x \).

- Base 2 is the most common for computer applications; denoted \( \lg x \).

**Convention**

Unless stated otherwise, all logarithms will be taken with base 2.

- The symbol \( \lg \) denotes a logarithm with base 2,
- and the symbol \( \ln \) denotes a natural logarithm.

If the base for logarithms is not specified or makes no difference, then the symbol \( \log \) will be used.

To convert logarithms from one base to another, multiply by a constant factor, the logarithm of the first base with respect to the second.

\[
\begin{align*}
\lg e & \approx 1.442695041, \\
\ln 2 & \approx 0.693147181, \\
\ln 10 & \approx 2.302585093.
\end{align*}
\]
Graphs of Logarithm Functions

Data Structures and Program Design In C++
Transp. 6, Sect. A.2, Logarithms

Log-Log Graph, Insertion and Merge Sorts

Comparisons of keys, average

Insertion sort

Merge sort
Harmonic Numbers

The \( n^{th} \) harmonic number is defined to be the sum

\[
H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}
\]

of the reciprocals of the integers from 1 to \( n \).

**Theorem A.4** The harmonic number \( H_n, n \geq 1 \), satisfies

\[
H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \epsilon,
\]

where \( 0 < \epsilon < 1/(252n^6) \), and \( \gamma \approx 0.577215665 \) is known as Euler’s constant.
Permutations and Combinations

A permutation of objects is an ordering or arrangement of the objects in a row.

<table>
<thead>
<tr>
<th>Objects to permute:</th>
<th>a b c d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Choose a first:</td>
<td>a b c d a b d c a c b d a c d b a d b c a d c b</td>
</tr>
<tr>
<td>Choose b first:</td>
<td>b a c d b a d c b c a d b c d a b d a c b d c a</td>
</tr>
<tr>
<td>Choose c first:</td>
<td>c a b d c a d b c b a d c b d a c d a b c d b a</td>
</tr>
<tr>
<td>Choose d first:</td>
<td>d a b c d a c b d b a c d b c a d c a b d c b a</td>
</tr>
</tbody>
</table>

A combination of \( n \) objects taken \( k \) at a time is a choice of \( k \) objects out of the \( n \), without regard for the order of selection. The number of such combinations is

\[
C(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

<table>
<thead>
<tr>
<th>Objects from which to choose:</th>
<th>a b c d e f</th>
</tr>
</thead>
<tbody>
<tr>
<td>a b c</td>
<td>a c d a d f b c f c d e</td>
</tr>
<tr>
<td>a b d</td>
<td>a c e a e f b d e c d f</td>
</tr>
<tr>
<td>a b e</td>
<td>a c f b c d b d f c e f</td>
</tr>
<tr>
<td>a b f</td>
<td>a d e b c e b e f d e f</td>
</tr>
</tbody>
</table>

The number of combinations \( C(n, k) \) is called a binomial coefficient, since it appears as the coefficient of \( x^k y^{n-k} \) in the expansion of \( (x + y)^n \).
Stirling’s Approximation to Factorials

The following results give good approximations to the factorial of a nonnegative integer, \( n! = n \times (n - 1) \times \cdots \times 1 \).

**Theorem A.5**

\[
 n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left[1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right].
\]

**Corollary A.6**

\[
 \ln n! = \left(n + \frac{1}{2}\right) \ln n - n + \frac{1}{2} \ln(2\pi) + \frac{1}{12n} + O\left(\frac{1}{n^2}\right).
\]
The Fibonacci numbers originated as an exercise in arithmetic proposed by Leonardo Fibonacci in 1202:

How many pairs of rabbits can be produced from a single pair in a year? We start with a single newly born pair; it takes one month for a pair to mature, after which they produce a new pair each month, and the rabbits never die.

The **Fibonacci numbers** are formally defined by the recurrence relation,

\[ F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2. \]

By the method of **generating functions**, the Fibonacci numbers are evaluated as

\[ F_n = \frac{1}{\sqrt{5}} (\phi^n - \psi^n), \]

where

\[ \phi = \frac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad \psi = 1 - \phi = \frac{1}{2}(1 - \sqrt{5}). \]

Approximate values for \( \phi \) and \( \psi \) are

\[ \phi \approx 1.618034 \quad \text{and} \quad \psi \approx -0.618034. \]

The absolute value of \( \psi \) is sufficiently small that \( F_n \) is always \( \phi^n / \sqrt{5} \) rounded to the nearest integer.
**DEFINITION** For \( n \geq 0 \), the \( n^{th} \) **Catalan number** is defined to be

\[
\text{Cat}(n) = \frac{C(2n, n)}{n + 1} = \frac{(2n)!}{(n + 1)!n!}.
\]

**THEOREM A.7** The number of distinct binary trees with \( n \) vertices, \( n \geq 0 \), is the \( n^{th} \) Catalan number \( \text{Cat}(n) \).

**LEMMA A.8** There is a one-to-one correspondence between the orchards with \( n \) vertices and the well-formed sequences of \( n \) left parentheses and \( n \) right parentheses, \( n \geq 0 \).

**LEMMA A.9** The sequences of \( n \) left and \( n \) right parentheses that are not well formed correspond exactly to all sequences of \( n - 1 \) left parentheses and \( n + 1 \) right parentheses (in all possible orders).

**COROLLARY A.10** The number of well-formed sequences of \( n \) left and \( n \) right parentheses, the number of permutations of \( n \) objects obtainable by a stack, the number of orchards with \( n \) vertices, and the number of binary trees with \( n \) vertices are all equal to the \( n^{th} \) Catalan number \( \text{Cat}(n) \).
Problems Related to Catalan Numbers

Bracketed form of orchards:

\[(a) \quad (a)\quad (a)(b) \quad (a(b)(c)(d)) \quad (a(b(c(d))))(e(f))\]

Triangulations of a hexagon:
Numerical Results

Approximation to Catalan numbers:

Stirling’s approximation gives:

\[ \text{Cat}(n) \approx \frac{4^n}{(n + 1)\sqrt{\pi n}} \]

The first twenty Catalan numbers:

<table>
<thead>
<tr>
<th>n</th>
<th>Cat(n)</th>
<th>n</th>
<th>Cat(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>10</td>
<td>16,796</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>11</td>
<td>58,786</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>12</td>
<td>208,012</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>13</td>
<td>742,900</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>14</td>
<td>2,674,440</td>
</tr>
<tr>
<td>5</td>
<td>42</td>
<td>15</td>
<td>9,694,845</td>
</tr>
<tr>
<td>6</td>
<td>132</td>
<td>16</td>
<td>35,357,670</td>
</tr>
<tr>
<td>7</td>
<td>429</td>
<td>17</td>
<td>129,644,790</td>
</tr>
<tr>
<td>8</td>
<td>1,430</td>
<td>18</td>
<td>477,638,700</td>
</tr>
<tr>
<td>9</td>
<td>4,862</td>
<td>19</td>
<td>1,767,263,190</td>
</tr>
</tbody>
</table>