

Paradoxes of Fuzzy Logic, Revisited

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This paper is a reply to the article entitled *Elkan's Theoretical Argument, Reconsidered* by Prof. Enric Trillas and Prof. Claudi Alsina. I would like to express my thanks to Dr. Piero Bonissone for inviting me to write this paper and for showing me the article by Trillas and Alsina in advance of its publication.

Ever since mathematical studies of fuzziness began with a 1965 paper entitled *Fuzzy Sets* by Lotfi Zadeh [Zad65], it has been important to distinguish between fuzzy logic and fuzzy set theory. The theorem in my paper *The Paradoxical Success of Fuzzy Logic*, as published in 1993 [Elk93] and also as revised in 1994 [Elk94], is a result about standard fuzzy logic. The paper of Trillas and Alsina concerns fuzzy sets, and therefore erroneously states that the theorem is false.

At the start of their Section 2, Trillas and Alsina write:

An equivalent formulation in Fuzzy Logic to what Professor Elkan claimed in references [8] and [9] is:

“If in the SFST $([0, 1]^X, \text{Min}, \text{Max}, 1 - id)$ the law

$$(A \cap B')' = B \cup (A' \cap B') \quad (1)$$

for any A, B in $[0, 1]^X$ is imposed, then either $A = B$ or $A = 1 - B$ ”

The statement quoted is not equivalent to the claim in my paper, because my theorem is not about fuzzy sets in $[0, 1]^X$. Instead, it is about something simpler but of greater practical importance: fuzzy truth values attached to individual logical assertions. In my paper, variables A, B , etc. refer to assertions in fuzzy logic that have truth values that are single numbers between 0 and 1.

Given the understanding that $A(x), B(x)$, etc. refer to the truth values of individual assertions, Theorem 1 and Corollary 1 of Trillas and Alsina are correct. The paragraph immediately after Corollary 1 is correct when talking about fuzzy sets, but irrelevant when talking only about individual fuzzy assertions, which is the context of my paper.

My theorem says that if we assume the equivalence $\neg(A \wedge \neg B) = B \vee (\neg A \wedge \neg B)$ is true for all sentences A and B in standard min/max fuzzy logic, then for any propositions P and Q it follows that the truth value of P is either the same as the truth value of Q , or the same as the truth value of $\neg Q$. The theorem can be proved using Corollary 1, by substituting P and Q , P and $\neg Q$, $\neg P$ and Q , etc. for A and B . As indicated by Trillas and Alsina at the start of their Section 2.2, the equivalence $\neg(A \wedge \neg B) = B \vee (\neg A \wedge \neg B)$ is not symmetric in A and B . The premise of my theorem is that the equivalence is true for any A and B , so also for B and A , etc.

At the end of their Section 2, Trillas and Alsina write

But in Elkan’s context, both statements “ $A \rightarrow B$ is the same as $B \rightarrow A$ ” and “the fuzzy sentences $B \cup (\neg A \cap \neg B)$ and $A \cup (\neg B \cap \neg A)$ are equal” appear to be incorrect.

These sentences, written in quotation marks by Trillas and Alsina exactly as shown above, do not appear in my paper; I never wrote them.

I did write “The theorem also applies to fuzzy set theory given the equation $\overline{A \cap B} = B \cup (\bar{A} \cap \bar{B})$, because Definition 1 can be understood as axiomatizing degrees of membership for fuzzy set intersections, unions, and complements.” This sentence says that the theorem is true pointwise for fuzzy sets A and B using the standard min, max, and $1 -$ operators. If A and B are sets in the space $[0, 1]^X$ then for any x in X , either $A(x) = B(x)$ or $A(x) = 1 - B(x)$ if the premise of the theorem is true. However, it does not follow that $A = B$ or $A = 1 - B$, because of course it may be the case that $A(x) = B(x)$ for x but $A(y) = 1 - B(y)$ for some y different from x . This rather obvious observation is the nub of Section 2 of the paper by Trillas and Alsina.

Sections 3, 4, and 5 of the paper by Trillas and Alsina concern generalized versions of fuzzy set theory that have nothing to do with my paper. Unfortunately they do not give any proof for their main result, Theorem 3, and no proof has been published elsewhere. The principal reference given by Trillas and Alsina for a proof, number 13, is described as “submitted” only.

Section 6 by Trillas and Alsina says “. . . Elkan was wrong in solving the equation

$$(A \cap B')' = B \cup (A' \cap B')$$

in the De Morgan’s Lattice $([0, 1]^X, \text{Min}, \text{Max}, 1 - id)$.” Actually, my paper never attempted to solve anything in this structure of fuzzy sets. Trillas and Alsina then write that my “paper opened a discussion that helped to shed light on some interesting issues, and we hope that the present paper will contribute to clarify some ideas . . .” However Trillas and Alsina do not discuss at all the second main paradox identified in my paper, which is the disconnect between fuzzy mathematics and fuzzy applications.

References

- [Elk93] Charles Elkan. The paradoxical success of fuzzy logic. In *Proceedings of the National Conference on Artificial Intelligence*, pages 698–703, Washington, D.C., August 1993.
- [Elk94] Charles Elkan. The paradoxical success of fuzzy logic. *IEEE Expert*, 6(6):3–8, August 1994. With fifteen responses on pp. 9–46 and a reply by the author on pp. 47–49.
- [Zad65] Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.