### Data Mining
Practical Machine Learning Tools and Techniques

Slides for Chapter 5 of *Data Mining* by I. H. Witten and E. Frank

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#### Credibility: Evaluating what’s been learned

- Issues: training, testing, tuning
- Predicting performance: confidence limits
- Holdout, cross-validation, bootstrap
- Comparing schemes: the t-test
- Predicting probabilities: loss functions
- Cost-sensitive measures
- Evaluating numeric prediction
- The Minimum Description Length principle

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#### Evaluation: the key to success

- How predictive is the model we learned?
- Error on the training data is *not* a good indicator of performance on future data
  - Otherwise 1-NN would be the optimum classifier!
- Simple solution that can be used if lots of (labeled) data is available:
  - Split data into training and test set
- However: (labeled) data is usually limited
  - More sophisticated techniques need to be used

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#### Issues in evaluation

- Statistical reliability of estimated differences in performance (→ significance tests)
- Choice of performance measure:
  - Number of correct classifications
  - Accuracy of probability estimates
  - Error in numeric predictions
- Costs assigned to different types of errors
  - Many practical applications involve costs

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#### Training and testing I

- Natural performance measure for classification problems: error rate
  - *Success*: instance’s class is predicted correctly
  - *Error*: instance’s class is predicted incorrectly
  - Error rate: proportion of errors made over the whole set of instances
- *Resubstitution error*: error rate obtained from training data
- Resubstitution error is (hopelessly) optimistic!

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#### Training and testing II

- *Test set*: independent instances that have played no part in formation of classifier
  - Assumption: both training data and test data are representative samples of the underlying problem
- Test and training data may differ in nature
  - Example: classifiers built using customer data from two different towns *A* and *B*
    - To estimate performance of classifier from town *A* in completely new town, test it on data from *B*
Note on parameter tuning

- It is important that the test data is not used in any way to create the classifier
- Some learning schemes operate in two stages:
  - Stage 1: build the basic structure
  - Stage 2: optimize parameter settings
- The test data can’t be used for parameter tuning!
- Proper procedure uses three sets: training data, validation data, and test data
- Validation data is used to optimize parameters

Making the most of the data

- Once evaluation is complete, all the data can be used to build the final classifier
- Generally, the larger the training data the better the classifier (but returns diminish)
- The larger the test data the more accurate the error estimate
- Holdout procedure: method of splitting original data into training and test set
  - Dilemma: ideally both training set and test set should be large!

Predicting performance

- Assume the estimated error rate is 25%. How close is this to the true error rate?
  - Depends on the amount of test data
- Prediction is just like tossing a (biased!) coin
  - “Head” is a “success”, “tail” is an “error”
- In statistics, a succession of independent events like this is called a Bernoulli process
  - Statistical theory provides us with confidence intervals for the true underlying proportion

Confidence intervals

- We can say: $p$ lies within a certain specified interval with a certain specified confidence
- Example: $S=750$ successes in $N=1000$ trials
  - Estimated success rate: 75%
  - How close is this to true success rate $p$?
    - Answer: with 80% confidence $p \in [73.2 , 76.7]$
- Another example: $S=75$ and $N=100$
  - Estimated success rate: 75%
  - With 80% confidence $p \in [69.1 , 80.1]$

Mean and variance

- Mean and variance for a Bernoulli trial: $p, p(1-p)$
  - Expected success rate $f=S/N$
  - Mean and variance for $f : p, p(1-p)/N$
  - For large enough $N, f$ follows a Normal distribution
  - $c\%$ confidence interval $[-z \leq X \leq z]$ for random variable with 0 mean is given by:
    - $Pr[-z \leq X \leq z] = c$
    - With a symmetric distribution:
      - $Pr[-z \leq X \leq z] = 1 - 2 \times Pr[X \geq z]$

Confidence limits

- Confidence limits for the normal distribution with 0 mean and a variance of 1:
  - $Pr[X \leq z] = z$
    - 0.1%: 3.09
    - 0.5%: 2.58
    - 1%: 2.33
    - 5%: 1.65
    - 10%: 1.28
    - 20%: 0.84
    - 40%: 0.25
- Thus:
  - $Pr[-1.65 \leq X \leq 1.65] = 90\%$
- To use this we have to reduce our random variable $f$ to have 0 mean and unit variance
Transforming $f$

- Transformed value for $f$:
  \[
  \frac{f-p}{\sqrt{p(1-p)/N}}
  \]
  (i.e. subtract the mean and divide by the standard deviation)

- Resulting equation:
  \[
  Pr[-z \leq \frac{f-p}{\sqrt{p(1-p)/N}} \leq z] = c
  \]
  \[
  p = \left( f + z^2 + z \sqrt{\frac{1}{N} - f^2 + \frac{z^2}{4N^2}} \right) / (1 + z^2)
  \]

Examples

- $f = 75\%, N = 1000, c = 80\%$ (so that $z = 1.28$):
  \[
  p \in [0.732, 0.767]
  \]
- $f = 75\%, N = 100, c = 80\%$ (so that $z = 1.28$):
  \[
  p \in [0.691, 0.801]
  \]
- Note that normal distribution assumption is only valid for large $N$ (i.e. $N > 100$)
- $f = 75\%, N = 10, c = 80\%$ (so that $z = 1.28$):
  \[
  p \in [0.549, 0.881]
  \]
  (should be taken with a grain of salt)

Holdout estimation

- What to do if the amount of data is limited?
- The holdout method reserves a certain amount for testing and uses the remainder for training
  - Usually: one third for testing, the rest for training
- Problem: the samples might not be representative
  - Example: class might be missing in the test data
- Advanced version uses stratification
  - Ensures that each class is represented with approximately equal proportions in both subsets

Repeated holdout method

- Holdout estimate can be made more reliable by repeating the process with different subsamples
  - In each iteration, a certain proportion is randomly selected for training (possibly with stratification)
  - The error rates on the different iterations are averaged to yield an overall error rate
- This is called the repeated holdout method
- Still not optimum: the different test sets overlap
  - Can we prevent overlapping?

Cross-validation

- Cross-validation avoids overlapping test sets
  - First step: split data into $k$ subsets of equal size
  - Second step: use each subset in turn for testing, the remainder for training
- Called $k$-fold cross-validation
- Often the subsets are stratified before the cross-validation is performed
- The error estimates are averaged to yield an overall error estimate

More on cross-validation

- Standard method for evaluation: stratified ten-fold cross-validation
- Why ten?
  - Extensive experiments have shown that this is the best choice to get an accurate estimate
  - There is also some theoretical evidence for this
- Stratification reduces the estimate’s variance
- Even better: repeated stratified cross-validation
  - E.g. ten-fold cross-validation is repeated ten times and results are averaged (reduces the variance)
**Leave-One-Out cross-validation**

- Leave-One-Out: a particular form of cross-validation:
  - Set number of folds to number of training instances
  - I.e., for n training instances, build classifier n times
- Makes best use of the data
- Involves no random subsampling
- Very computationally expensive
  - (exception: NN)

**Leave-One-Out-CV and stratification**

- Disadvantage of Leave-One-Out-CV: stratification is not possible
  - It guarantees a non-stratified sample because there is only one instance in the test set!
- Extreme example: random dataset split equally into two classes
  - Best inducer predicts majority class
  - 50% accuracy on fresh data
  - Leave-One-Out-CV estimate is 100% error!

**The bootstrap**

- CV uses sampling without replacement
  - The same instance, once selected, can not be selected again for a particular training/test set
- The bootstrap uses sampling with replacement to form the training set
  - Sample a dataset of n instances n times with replacement to form a new dataset of n instances
  - Use this data as the training set
  - Use the instances from the original dataset that don’t occur in the new training set for testing

**The 0.632 bootstrap**

- Also called the 0.632 bootstrap
  - A particular instance has a probability of $1 - \frac{1}{n}$ of not being picked
  - Thus its probability of ending up in the test data is: $\left(1 - \frac{1}{n}\right)^n \approx e^{-1} \approx 0.368$
  - This means the training data will contain approximately 63.2% of the instances

**Estimating error with the bootstrap**

- The error estimate on the test data will be very pessimistic
  - Trained on just ~63% of the instances
- Therefore, combine it with the resubstitution error:
  $$ err = 0.632 \times e_{\text{test instances}} + 0.368 \times e_{\text{training instances}} $$
- The resubstitution error gets less weight than the error on the test data
- Repeat process several times with different replacement samples; average the results

**More on the bootstrap**

- Probably the best way of estimating performance for very small datasets
- However, it has some problems
  - Consider the random dataset from above
  - A perfect memorizer will achieve 0% resubstitution error and ~50% error on test data
  - Bootstrap estimate for this classifier:
    $$ err = 0.632 \times 50% + 0.368 \times 0% = 31.6% $$
  - True expected error: 50%
Counting the cost

- In practice, different types of classification errors often incur different costs
- Examples:
  - Terrorist profiling
  - "Not a terrorist" correct 99.99% of the time
  - Loan decisions
  - Oil-slick detection
  - Fault diagnosis
  - Promotional mailing

The kappa statistic

- Two confusion matrices for a 3-class problem: actual predictor (left) vs. random predictor (right)
- Number of successes: sum of entries in diagonal (D)
- Kappa statistic: \( D_{observed} - D_{expected} \)
- measures relative improvement over random predictor

More measures...

- Percentage of retrieved documents that are relevant: \( precision = \frac{TP}{TP+FP} \)
- Percentage of relevant documents that are returned: \( recall = \frac{TP}{TP+FN} \)
- Precision/recall curves have hyperbolic shape
- Summary measures: average precision at 20%, 50% and 80% recall (three-point average recall)
- \( F\)-measure = \( \frac{2 \times recall \times precision}{recall + precision} \)
- \( sensitivity \times specificity = \frac{(TP / (TP + FP)) \times (TN / (TP + TN))} \)
- Area under the ROC curve (AUC): probability that randomly chosen positive instance is ranked above randomly chosen negative one
Evaluating numeric prediction

- Same strategies: independent test set, cross-validation, significance tests, etc.
- Difference: error measures
- Actual target values: $a_1, a_2, \ldots, a_n$
- Predicted target values: $p_1, p_2, \ldots, p_n$
- Most popular measure: mean-squared error
  \[
  \frac{1}{n} \sum_{i=1}^{n} (p_i - a_i)^2
  \]
- Easy to manipulate mathematically

Improvement on the mean

- How much does the scheme improve on simply predicting the average?

  - The relative squared error is:
    \[
    \frac{\sum (p_i - a_i)^2}{\sum (\bar{a} - a_i)^2}
    \]
  
  - The relative absolute error is:
    \[
    \frac{\sum |p_i - a_i|}{\sum |\bar{a} - a_i|}
    \]

Other measures

- The root mean-squared error:
  \[
  \sqrt{\frac{1}{n} \sum_{i=1}^{n} (p_i - a_i)^2}
  \]
- The mean absolute error is less sensitive to outliers than the mean-squared error:
  \[
  \frac{\sum |p_i - a_i|}{n}
  \]
- Sometimes relative error values are more appropriate (e.g. 10% for an error of 50 when predicting 5000)

Correlation coefficient

- Measures the statistical correlation between the predicted values and the actual values
  \[
  S_{PA} = \frac{\sum (p_i - \bar{p})(a_i - \bar{a})}{\sqrt{S_P S_A}}
  \]
  \[
  S_P = \frac{\sum (p_i - \bar{p})^2}{n-1}
  \]
  \[
  S_A = \frac{\sum (a_i - \bar{a})^2}{n-1}
  \]
- Scale independent, between -1 and +1
- Good performance leads to large values!
Cost-sensitive classification

- Can take costs into account when making predictions
  - Basic idea: only predict high-cost class when very confident about prediction
- Given: predicted class probabilities
  - Normally we just predict the most likely class
  - Here, we should make the prediction that minimizes the expected cost
  - Expected cost: dot product of vector of class probabilities and appropriate column in cost matrix
  - Choose column (class) that minimizes expected cost

Cost-sensitivity learning

- So far we haven’t taken costs into account at training time
- Most learning schemes do not perform cost-sensitive learning
  - They generate the same classifier no matter what costs are assigned to the different classes
  - Example: standard decision tree learner
- Simple methods for cost-sensitive learning:
  - Resampling of instances according to costs
  - Weighting of instances according to costs
  - Some schemes can take costs into account by varying a parameter, e.g., naïve Bayes

Cost Matrix

Example:
Let cost of carrying umbrella = 1

<table>
<thead>
<tr>
<th>Predicted</th>
<th>Actual</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>0</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Rain</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Formalize the idea that mistakes have different costs

Cost Matrix

Example:
Let cost of carrying umbrella = 1

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<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Assume for now we know probability of sun vs. rain

\( P_{\text{Rain}} = 1 - P_{\text{Sun}} \)
Making Cost-Optimal Decisions

Example:
Let cost of carrying umbrella = 1

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
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<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

1. Find expected cost of predicting sun
2. Find expected cost of predicting rain
3. Predict which ever has a lower expected cost

Calculating Expected Costs

Worked Example:
Let cost of carrying umbrella = 1

<table>
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<tr>
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<th>Actual</th>
<th>Predicted</th>
</tr>
</thead>
<tbody>
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<td>10</td>
</tr>
<tr>
<td>Rain</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

1. Expected cost of predicting ‘Sun’
   \[ P_{\text{Sun}} \times 1 + 10 \times P_{\text{Rain}} \]

2. Expected cost of predicting ‘Rain’
   \[ 0 \times P_{\text{Sun}} + 1 \times P_{\text{Rain}} \]

3. When should we predict ‘Rain’?
   \[
   \text{Cost}_{\text{Sun}} < \text{Cost}_{\text{Rain}}
   \]
   \[
   1 < 10 \times P_{\text{Rain}}
   \]
   \[
   1/10 < P_{\text{Rain}}
   \]

Cost Matrix: Results

<table>
<thead>
<tr>
<th></th>
<th>Actual</th>
<th>Predicted</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Rain</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

- Take umbrella even if \( P_{\text{Rain}} \) as low as 10%
- Might act as if one class is true even if another is more probable

Cost Matrix: Summary

\[
E(X) = \sum P(X) \times X
\]

- Expected cost if predict \( A \)
\[
P(\text{Actual class} = A) \times c_{00} + P(\text{Actual class} = B) \times c_{01}
\]

vs.

- Expected cost if predict \( B \)
\[
P(\text{Actual class} = A) \times c_{10} + P(\text{Actual class} = B) \times c_{11}
\]

In WEKA's Explorer GUI

- cost matrix
  - supplied as parameter or loaded from a file

- weka.classifiers.meta.CostSensitiveClassifier
  - cost-sensitive classification
    - predicts the class with the least expected misclassification cost rather than the most likely one
  - cost-sensitive learning
    - reweights training instances according to the total cost assigned to each class
**Lift charts**

- In practice, costs are rarely known
- Decisions are usually made by comparing possible scenarios
- Example: promotional mailout to 1,000,000 households
  - Mail to all; 0.1% respond (1000)
  - Data mining tool identifies subset of 100,000 most promising, 0.4% of these respond (400)
    - 40% of responses for 10% of cost may pay off
  - Identify subset of 400,000 most promising, 0.2% respond (800)
- A lift chart allows a visual comparison

**Generating a lift chart**

- Sort instances according to predicted probability of being positive:
  
<table>
<thead>
<tr>
<th>Predicted probability</th>
<th>Actual class</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Yes</td>
</tr>
<tr>
<td>2</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>No</td>
</tr>
<tr>
<td>4</td>
<td>Yes</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

  - x axis is sample size
  - y axis is number of true positives

**A hypothetical lift chart**

![Lift Chart Diagram](chart.png)

- 40% of responses for 10% of cost
- 80% of responses for 40% of cost

**ROC curves**

- ROC curves are similar to lift charts
  - Stands for “receiver operating characteristic”
  - Used in signal detection to show tradeoff between hit rate and false alarm rate over noisy channel
- Differences to lift chart:
  - y axis shows percentage of true positives in sample rather than absolute number
  - x axis shows percentage of false positives in sample rather than sample size

**A sample ROC curve**

![ROC Curve Diagram](curve.png)

- Jagged curve—one set of test data
- Smooth curve—use cross-validation

**Cross-validation and ROC curves**

- Simple method of getting a ROC curve using cross-validation:
  - Collect probabilities for instances in test folds
  - Sort instances according to probabilities
- This method is implemented in WEKA
- However, this is just one possibility
  - Another possibility is to generate an ROC curve for each fold and average them
In WEKA’s Explorer GUI

- use threshold curves to generate ROC curves

ROC curves for two schemes

- For a small, focused sample, use method A
- For a larger one, use method B
- In between, choose between A and B with appropriate probabilities

The convex hull

- Given two learning schemes we can achieve any point on the convex hull!
- TP and FP rates for scheme 1: \( t_1 \) and \( f_1 \)
- TP and FP rates for scheme 2: \( t_2 \) and \( f_2 \)
- If scheme 1 is used to predict \( 100 \times q \% \) of the cases and scheme 2 for the rest, then
  - TP rate for combined scheme: \( q \times t_1 + (1-q) \times t_2 \)
  - FP rate for combined scheme: \( q \times f_1 + (1-q) \times f_2 \)

Cost curves

- Cost curves plot expected costs directly
- Example for case with uniform costs (i.e. error):

Summary of some measures

<table>
<thead>
<tr>
<th>Domain</th>
<th>Plot</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lift chart</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ROC curve</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Recall-precision curve</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Marketing</td>
<td>TP Subset size</td>
<td>TP (TP+FP)/(TP+FP+TN+FN)</td>
</tr>
<tr>
<td>Communications</td>
<td>TP rate</td>
<td>TP/(TP+FN)</td>
</tr>
<tr>
<td>Information retrieval</td>
<td>Recall Precision</td>
<td>TP/(TP+FN)</td>
</tr>
</tbody>
</table>

Data Mining: Practical Machine Learning Tools and Techniques (Chapter 5)
Cost curves: example with costs

Probability cost function \(p_c(+) = \frac{p_c(+|+)}{p_c(+|+)}\)
Normalized expected cost = \(fn \times p_c(+) + fp \times (1 - p_c(+)\)

Comparing schemes I

- Frequent question: which of two learning schemes performs better?
- Note: this is domain dependent!
- Obvious way: compare 10-fold CV estimates
- Generally sufficient in applications (we don’t loose if the chosen method is not truly better)
- However, what about machine learning research?
  - Need to show convincingly that a particular method works better

Comparing schemes II

- Want to show that scheme A is better than scheme B in a particular domain
  - For a given amount of training data
  - On average, across all possible training sets
- Let’s assume we have an infinite amount of data from the domain:
  - Sample infinitely many dataset of specified size
  - Obtain cross-validation estimate on each dataset for each scheme
  - Check if mean accuracy for scheme A is better than mean accuracy for scheme B

Paired t-test

- In practice we have limited data and a limited number of estimates for computing the mean
- Student’s t-test tells whether the means of two samples are significantly different
- In our case the samples are cross-validation estimates for different datasets from the domain
- Use a paired t-test because the individual samples are paired
  - The same CV is applied twice

Distribution of the means

- \(x_1, x_2, \ldots, x_k\) and \(y_1, y_2, \ldots, y_k\) are the 2k samples for the k different datasets
- \(m_x\) and \(m_y\) are the means
- With enough samples, the mean of a set of independent samples is normally distributed
- Estimated variances of the means are \(\sigma^2\) and \(\sigma^2\)
- If \(\mu_x\) and \(\mu_y\) are the true means then
  \[\frac{m_x - \mu_x}{\sigma^2/k} \quad \frac{m_y - \mu_y}{\sigma^2/k}\]
  are approximately normally distributed with mean 0, variance 1
Student’s distribution

- With small samples ($k < 100$) the mean follows
  Student’s distribution with $k-1$ degrees of freedom

- Confidence limits:

<table>
<thead>
<tr>
<th>9 degrees of freedom distribution</th>
<th>normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(Z &lt; z)$</td>
<td>$z$</td>
</tr>
<tr>
<td>0.1%</td>
<td>4.30</td>
</tr>
<tr>
<td>0.5%</td>
<td>3.25</td>
</tr>
<tr>
<td>1%</td>
<td>2.82</td>
</tr>
<tr>
<td>5%</td>
<td>1.83</td>
</tr>
<tr>
<td>10%</td>
<td>1.58</td>
</tr>
<tr>
<td>20%</td>
<td>0.88</td>
</tr>
</tbody>
</table>

Assuming we have 10 estimates

Distribution of the differences

- Let $m_d = m_1 - m_2$
  - The difference of the means ($m_j$) also has a Student’s distribution with $k-1$ degrees of freedom
  - Let $\sigma^2$ be the variance of the difference
  - The standardized version of $m_d$ is called the t-statistic:
    \[ t = \frac{m_d}{\sqrt{\frac{\sigma^2}{k}} + \frac{\sigma^2}{j}} \]
  - We use $t$ to perform the t-test

Performing the test

- Fix a significance level
  - If a difference is significant at the $\alpha$ level, there is a (100-$\alpha$)% chance that the true means differ
  - Divide the significance level by two because the test is two-tailed
    - i.e. the true difference can be +ve or – ve
  - Look up the value for $z$ that corresponds to $\alpha/2$
- If $t \leq z$ or $t \geq z$ then the difference is significant
  - i.e. the null hypothesis (that the difference is zero) can be rejected

Unpaired observations

If the CV estimates are from different datasets, they are no longer paired (or maybe we used $k$-fold CV for one scheme, and $j$-fold CV for the other one)

- Then we have to use an unpaired $t$-test with $\min(k,j) - 1$ degrees of freedom
  - The $t$-statistic becomes:
    \[ t = \frac{m_d}{\sqrt{\frac{\sigma^2}{k} + \frac{\sigma^2}{j}}} \]

Dependent estimates

- We assumed that we have enough data to create several datasets of the desired size
- Need to re-use data if that’s not the case
  - E.g. running cross-validations with different randomizations on the same data
  - Samples become dependent $\Rightarrow$ insignificant differences can become significant
- A heuristic test is the corrected resampled $t$-test:
  - Assume we use the repeated hold-out method, with $n_1$ instances for training and $n_2$ for testing
  - New test statistic is:
    \[ t = \frac{m_d}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \]
Predicting probabilities

- Performance measure so far: success rate
- Also called 0-1 loss function:
  - “loss” is zero if the prediction is correct or one if it is not
- Most classifiers produce class probabilities
- Depending on the application, we might want to check the accuracy of the probability estimates
- 0-1 loss is not the right thing to use in those cases

Quadratic loss function

- $p_1, \ldots, p_n$ are probability estimates for an instance
- $c$ is the index of the instance's actual class
- $a_1, \ldots, a_n = 0$, except for $a_c$, which is 1
- Quadratic loss is: $\sum_j (p_j - a_j)^2 = \sum_{j \neq c} p_j^2 + (a - p_c)^2$
- Want to minimize $E(\sum_j (p_j a_j))$
- Can show that this is minimized when $p_j = p_j^*$, the true probabilities
Informational loss function

- The informational loss function is \(-\log(p_c)\), where \(c\) is the index of the instance's actual class
- Number of bits required to communicate the actual class
- Let \(p_1' \ldots p_k'\) be the true class probabilities
- Then the expected value for the loss function is:
  \[-p_1' \log_2 p_1 - \ldots - p_k' \log_2 p_k\]
- Justification: minimized when \(p_i = p_j\)
- Difficulty: zero-frequency problem

Discussion

- Which loss function to choose?
  - Both encourage honesty
  - Quadratic loss function takes into account all class probability estimates for an instance
  - Informational loss focuses only on the probability estimate for the actual class
  - Quadratic loss is bounded: \(1 + \sum_j p_j^2\)
  - Informational loss can be infinite
- Informational loss is related to MDL principle [later]

The MDL principle

- MDL stands for minimum description length
- The description length is defined as:
  space required to describe a theory +
  space required to describe the theory's mistakes
- In our case the theory is the classifier and the mistakes are the errors on the training data
- Aim: we seek a classifier with minimal DL
- MDL principle is a model selection criterion

Model selection criteria

- Model selection criteria attempt to find a good compromise between:
  - The complexity of a model
  - Its prediction accuracy on the training data
  - Reasoning: a good model is a simple model that achieves high accuracy on the given data
- Also known as Occam’s Razor: the best theory is the smallest one that describes all the facts

William of Ockham, born in the village of Ockham in Surrey (England) about 1285, was the most influential philosopher of the 14th century and a controversial theologian.

Elegance vs. errors

- Theory 1: very simple, elegant theory that explains the data almost perfectly
- Theory 2: significantly more complex theory that reproduces the data without mistakes
- Theory 1 is probably preferable
- Classical example: Kepler's three laws on planetary motion
  - Less accurate than Copernicus’s latest refinement of the Ptolemaic theory of epicycles

MDL and compression

- MDL principle relates to data compression:
  - The best theory is the one that compresses the data the most
  - I.e. to compress a dataset we generate a model and then store the model and its mistakes
- We need to compute
  - (a) size of the model, and
  - (b) space needed to encode the errors
- (b) easy: use the informational loss function
- (a) need a method to encode the model
### MDL and Bayes’s theorem

- $L[T]$ = “length” of the theory
- $L[E|T]$ = training set encoded wrt the theory
- Description length = $L[T] + L[E|T]$
- Bayes’s theorem gives a *posteriori* probability of a theory given the data:
  
  $$Pr[T|E] = \frac{Pr[E|T]Pr[T]}{Pr[E]}$$

- Equivalent to:
  
  $$-\log Pr[T|E] = -\log Pr[E|T] - \log Pr[T] + \frac{\log Pr[E]}{\text{constant}}$$

### MDL and MAP

- MAP stands for *maximum a posteriori probability*
- Finding the MAP theory corresponds to finding the MDL theory
- Difficult bit in applying the MAP principle:
  determining the prior probability $Pr[T]$ of the theory
- Corresponds to difficult part in applying the MDL principle: coding scheme for the theory
- I.e. if we know a priori that a particular theory is more likely we need fewer bits to encode it

### Discussion of MDL principle

- Advantage: makes full use of the training data when selecting a model
- Disadvantage 1: appropriate coding scheme/prior probabilities for theories are crucial
- Disadvantage 2: no guarantee that the MDL theory is the one which minimizes the expected error
- Note: Occam’s Razor is an axiom!
- Epicurus’s *principle of multiple explanations*: keep all theories that are consistent with the data

### MDL and clustering

- Description length of theory:
  bits needed to encode the clusters
  - e.g. cluster centers
- Description length of data given theory:
  encode cluster membership and position relative to cluster
  - e.g. distance to cluster center
- Works if coding scheme uses less code space for small numbers than for large ones
- With nominal attributes, must communicate probability distributions for each cluster