CHAPTER 1

THE FUZZY WORLD

What's the process of parallel parking a car?

First you line up your car next to the one in front of your space. Then you angle the car back into the space, turning the steering wheel slightly to adjust your angle as you get closer to the curb. Now turn the wheel to back up straight and—nothing. Your rear tire's wedged against the curb.

OK. Go forward slowly, steering toward the curb until the rear tire straightens out. Fine—except, you're too far from the curb. Drive back and forth again, using shallower angles.

Now straight forward. Good, but a little too close to the car ahead. Back up a few inches. Thunk! Oops, that's the bumper of the car in back. Forward just a few inches. Stop! Perfect!! Congratulations. You've just parallel-parked your car.

And you've just performed a series of fuzzy operations.

Not fuzzy in the sense of being confused. But fuzzy in the real-world sense, like "going forward slowly" or "a bit hungry" or "partly cloudy"—the distinctions that people use in decision-making all the time, but that computers and other advanced technology haven't been able to handle.

What kind of problems? For one, waiting for an elevator at lunch hour. How do you program elevators so that they pick up the most people in the least amount of time? Or how do you program elevators to minimize the waiting time for the most people?

Suppose you're operating an automated subway system. How do you program a train to start up and slow down at stations so smoothly that the passengers hardly notice?

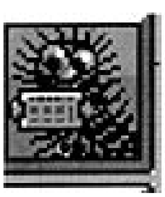
For that matter, how can you program a brake system on an automobile so that it works efficiently, taking road and tire conditions into account?

Perhaps you have a manufacturing process that requires a very steady temperature over a many hours. What's the most efficient and reliable method for achieving it?

Or, suppose you're filming an unpredictable and fast-moving event with your camcorder—say, a birthday party of 10 three-year-olds. What kind of a camera lets you move with the action and still end up with a very nonjerky image when you play it back?

Or, take a problem far from the realm of manufacturing and engineering, such as, how do you define the term *family* for the purposes of inclusion in health insurance policy?

Do all these situations have something in common? For one thing, they're all complex and dynamic. Also, like parallel parking, they're more easily characterized by words and shades of meaning than by mathematics.



In this book you'll be immersed in the fuzzy world, not an easy process. You'll meet the basics, manipulate the tools (simple and complex), and use them to solve real-world problems. You can make your experience interactive and hands on with a series of programs on the accompanying disk. (See the Preface for an explanation of how to load it onto your hard disk.) To make the trip easier, you'll be following in the many footsteps of our fuzzy field guide, Dr. Fuzzy. The good doctor will be on call through Help menus and will show up in the book chapters with hints, further information, and encouraging messages.

E-MAIL FROM DR. FUZZY

The real world is up and down, constantly moving and changing, and full of surprises. In other words, fuzzy.

Fuzzy techniques let you successfully handle real-world situations.

APPLES, ORANGES, OR IN BETWEEN?

As the fiber-conscious Dr. Fuzzy has discovered, one of the easiest ways to step into the fuzzy world is with a simple device found in most homes—a bowl of fruit. Conventional computers and simple digital control systems follow the either-or system. The digit's either zero or one. The answer's either yes or no. And the fruit bowl (or database cell) contains either apples or oranges.

Take Figure 1.1, for example. Is this a bowl of oranges? The answer is No.

How about Figure 1.2? Is it a bowl of oranges? The answer in this case is Yes.

This is an example of crisp logic, adequate for a situation in which the bowl does contain *either* totally apples *or* totally oranges. But life is often more complex. Take the case of the bowl in Figure 1.3. Someone has made a switch,

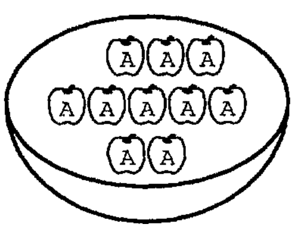


Figure 1.1: Is this a bowl of oranges?

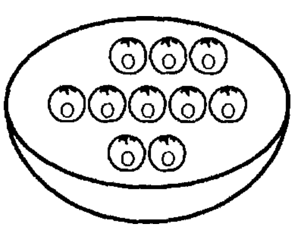


Figure 1.2: Is *this* a bowl of oranges?

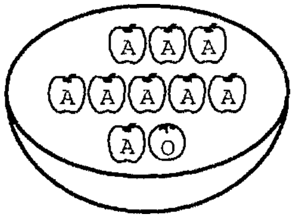


Figure 1.3: "Thinking fuzzy" about a bowl of oranges.

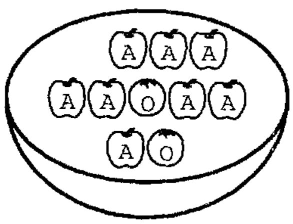


Figure 1.4: Fuzzy bowl of apples.

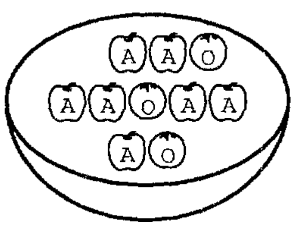


Figure 1.5: Fuzzy bowl of apples (continued).

swapping an orange for one of the apples in the Yes—Apple bowl. Is it a bowl of oranges?

Suppose another apple disappears, only to be replaced by an orange (Figure 1.4). The same thing happens again (Figure 1.5). And again (Figure 1.6). Is the bowl now a bowl of oranges? Suppose the process continues

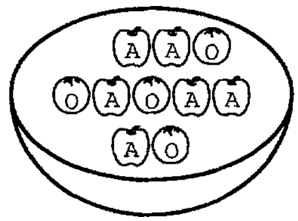


Figure 1.6: Fuzzy bowl of apples (continued).

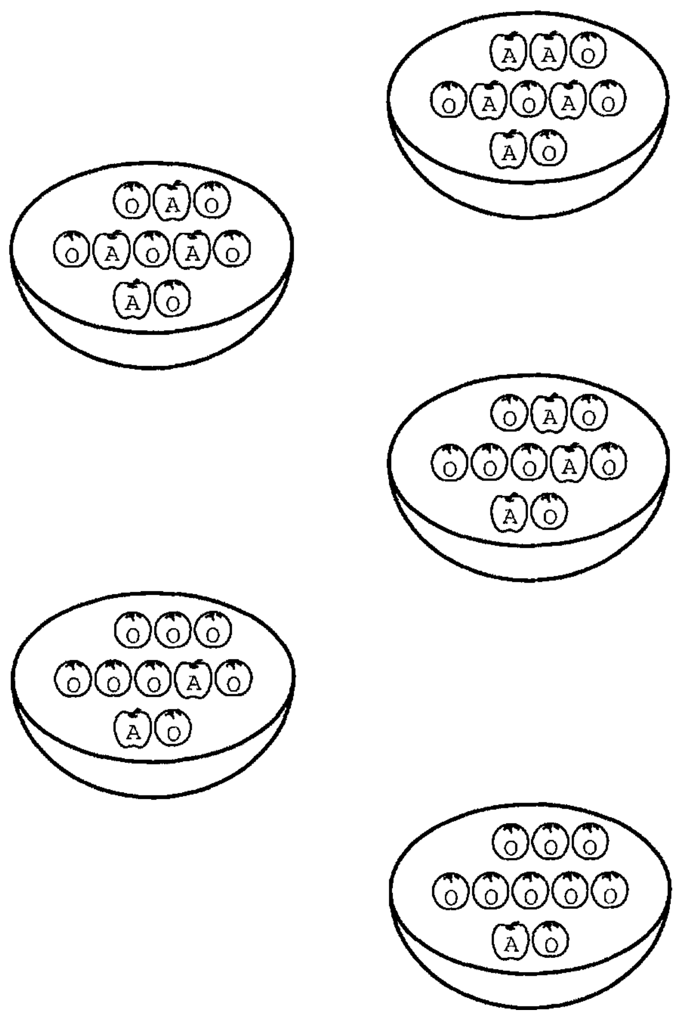


Figure 1.6: Fuzzy bowl of apples (continued).

(Figure 1.7). At some point, can you say that the "next bowl" contains oranges rather than apples?

This isn't a situation where you're unable to say Yes or No because you need more information. You have all the information you need. The situation itself makes *either* Yes *or* No inappropriate. In fact, if you had to say Yes or No, your answer would be less precise that if you answered One, or Some, or A Few, or Mostly—all of which are fuzzy answers, somewhere in between Yes and No. They handle the actual ambiguity in descriptions or presentations of reality.

Other ambiguities are possible. For example, if the apples were coated with orange candy, in which case the answer might be Maybe. The complexity of reality leads to truth being stranger than fiction. Fuzzy logic holds that crisp (0/1) logic is often a fiction. Fuzzy logic actually contains crisp logic as an extreme.

E-MAIL FROM DR. FUZZY

Really want to think fuzzy apples and oranges? They have less distinct boundaries than you might think.

Both apples and oranges are spheres, and both are about the same size. Both grow on trees that reproduce similarly. You can make a tasty drink from each. They even go to their rewards the same way, by being eaten and digested by people, or by being composted by my relatives, near and distant. If the apples are red, even the colors are related—

red + yellow = orange

And don't neglect the bowl. Both fruits nestle the same way in the same kind of bowl, and they leave similar amounts of unoccupied space.

With fuzzy logic the answer is Maybe, and its value ranges anywhere from 0 (No) to 1 (Yes).

E-MAIL FROM DR. FUZZY	Crisp sets handle only 0s and 1s. Fuzzy sets handle all values between 0 and 1.

Crisp

No Yes

Fuzzy

No Slightly Somewhat SortOf A Few Mostly Yes, Absolutely

Looking at the fruit bowls again (Figure 1.8), you might assign these fuzzy values to answer the question, Is this a bowl of oranges?

E-MAIL FROM DR. FUZZY	Characteristics of fuzziness: • Word based, not number based. For instance, *hot;* not *85*°. • Nonlinear changeable. • Analog (ambiguous), not digital (Yes/No).

If you really look at the way we make decisions, even the way we use computers and other machines, it's surprising that fuzziness isn't considered the *ordinary* way of functioning. Why isn't it? It all started with Aristotle (and his buddies).

IS THERE LIFE BEYOND MATH?

The either-apples-or-oranges system is known as "crisp" logic. It's the logic developed by the fourth century B.C. Greek philosopher Aristotle and is often called *Arisfotelian* in his honor. Aristotle got his idea from the work of an earlier Greek philosopher, Pythagoras, and his followers, who believed that matter was essentially numerical and that the universe could be defined as numerical relationships. Their work is traditionally credited with providing

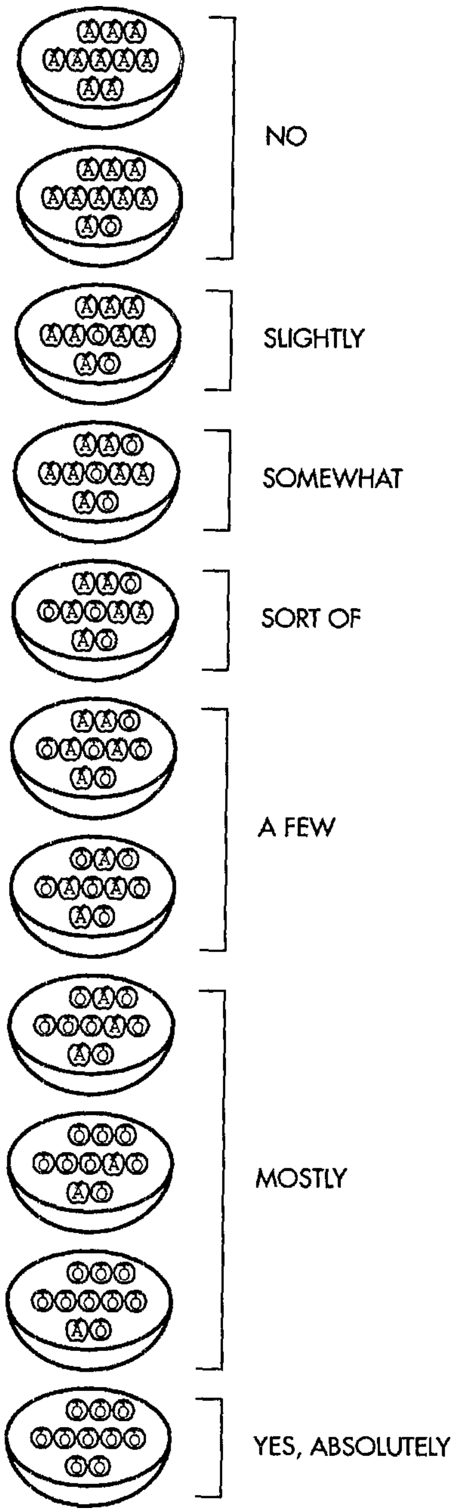


Figure 1.8: Fuzzy values.

the foundation of geometry and Western music (through the mathematics of tone relationships).

Aristotle extended the Pythagorean belief to the way people think and make decisions by allying the precision of math with the search for truth. By the tenth century A.D., Aristotelian logic was the basis of European and Middle Eastern thought. It has persisted for two reasons—it simplifies thinking about problems and makes "certainty" (or "truth") easier to prove and accept.

Vague Is Better

In 1994 fuzziness is the state of the art, but the idea isn't new by any means. It's gone under the name fuzzy for 25 years, but its roots go back 2,500 years. Even Aristotle considered that there were degrees of true-false, particularly in making statements about possible future events. Aristotle's teacher, Plato, had considered degrees of membership. In fact, the word *Platonic* embodies his concept of an intellectual ideal—for instance, of a chair—that could be realized only partially in human or physical terms. But Plato rejected the notion.

Skip to eighteenth century Europe, when three of the leading philosophers played around with the idea. The Irish philosopher and clergyman George Berkeley and the Scot David Hume thought that each concept has a concrete core, to which concepts that resemble it in some way are attracted. Hume in particular believed in the logic of common sense—reasoning based on the knowledge that ordinary people acquire by living in the world.

In Germany, Immanuel Kant considered that only mathematics could provide clean definitions, and many contradictory principles could not be resolved. For instance, matter could be divided infinitely, but at the same time could not be infinitely divided.

That particularly American school of philosophy called *pragmatism* was founded in the early years of this century by Charles Sanders Peirce, who stated that an idea's meaning is found in its consequences. Peirce was the first to consider "vagueness," rather than true-false, as a hallmark of how the world and people function.

The idea that "crisp" logic produced unmanageable contradictions was picked up and popularized at the beginning of the twentieth century by the flamboyant English philosopher and mathematician, Bertrand Russell.

He also studied the vagueness of language, as well as its precision, concluding that vagueness is a matter of degree.

E-MAIL FROM DR. FUZZY

Crisp logic has always had fuzzy edges in the form of paradoxes. One example is the apples-oranges question earlier in the chapter. Here are some ancient Greek versions:

- How many individual grains of sand can you remove from a sandpile before it isn't a pile any more (Zeno's paradox)?
- How many individual hairs can fall from a man's head before he becomes bald (Bertrand Russell's paradox)?

In ancient, politically incorrect mainland Greece they said, "All Cretans are liars. When a Cretan says that he's lying, is he telling the truth?" The logical problem: How stable is the idea of truth and falsity?

In the early twentieth century, Bertrand Russell (who seemed to be amazingly interested in human fuzz) asked: A man who's a barber advertises "I shave all men and only those who don't shave themselves." Who shaves the barber?

The down-home illustration involved this logical question: Can a set contain itself?

The German philosopher Ludwig Wittgenstein studied the ways in which a word can be used for several things that really have little in common, such as a *game,* which can be competitive or noncompetitive.

The original (0 or 1) set theory was invented by the nineteenth century German mathematician Georg Kantor. But this "crisp" set has the same shortcomings as the logic it's based on. The first logic of vagueness was developed in 1920 by the Polish philosopher Jan Lukasiewicz. He devised sets with possible membership values of 0, 1/2, and 1, later extending it by allowing an infinite number of values between 0 and 1.

Later in the twentieth century, the nature of mathematics, real-life events, and complexity all played roles in the examination of crispness. So did the amazing discovery of physicists such as Albert Einstein (relativity) and Werner Heisenberg (uncertainty). Einstein was quoted as saying, "As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."

The next big step forward came in 1937, at Cornell University, where Max Black considered the extent to which objects were members of a set, such as a chairlike object in the set Chair. He measured membership in degrees of usage and advocated a general theory of "vagueness."

The work of these nineteenth and twentieth century thinkers provided the grist for the mental mill of the founder of fuzzy logic, an American named Lotfi Zadeh.

Discovering Fuzziness

In the 1960s, Lotfi Zadeh invented fuzzy logic, which combines the concepts of crisp logic and the Lukasiewicz sets by defining graded membership. One of Zadeh's main insights was that mathematics can be used to link language and human intelligence. Many concepts are better defined by words than by mathematics, and fuzzy logic and its expression in fuzzy sets provide a discipline that can construct better models of reality.

E-MAIL FROM DR. FUZZY

Lotfi Zadeh says that fuzziness involves possibilities. For instance, it's possible that 6 is a large number, while it's impossible that 1 or 2 are large numbers. In this case, a fuzzy set of possible large numbers includes 3, 4, 5, and 6.

Daniel Schwartz, an American fuzzy logic researcher, organized fuzzy words under several headings. *Quantification* terms include all, most, many, about half, few, and no. *Usuality* includes always, frequently, often, occasionally, seldom, and never. *Likelihood* terms are certain, likely, uncertain, unlikely, and certainly not.

How do you think fuzzy" about a fuzzy word–also called a *linguistic variable*–in contrast to "thinking crisp"? Dimiter Driankov and several colleagues in Germany have pointed out three ways that highlight the difference.

Suppose the variable is *largeness*. Someone gives you the number 6 and says, "6 is a large number. Do you agree or disagree?"

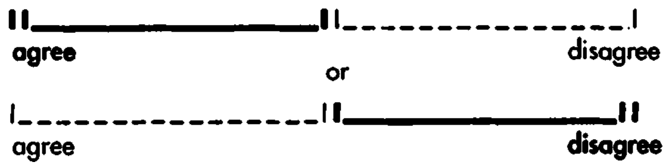


Figure 1.9: A threshold person either agrees or disagrees.

If you're a *threshold person, you* will flatly state either "I agree" or "I disagree." This can be drawn as in Figure 1.9.

An *estimator* will take a different approach, saying "I agree partially" (Figure 1.10). The answer may depend on the context in which the question is asked. The person might partly agree that 6 is a large number if the next number is 0.05. But if the next one is 50, then the person might disagree partially or totally.

A *conservative* takes still another approach, possibly saying, "I agree," "I disagree," or "I'm not sure." Public opinion polls often use this method. For instance, if the statement is "Are you willing to pay higher taxes to build more playgrounds"? Someone might answer, "I am if the playgrounds will help reduce juvenile crime."

Are any of these answers fuzzy? The threshold person has given a crisp answer–all or nothing. The other two people have given fuzzy ones. The estimator's answer involves a degree, so that there can be as many different responses as there are people answering the question. The conservative's answer recognizes that some questions by their nature may always have uncertain aspects or involve balancing tradeoffs.

Figure 1.10: An estimator may agree partially.

THE USES OF FUZZY LOGIC

Fuzzy systems can be used for estimating, decision-making, and mechanical control systems such as air conditioning, automobile controls, and even "smart" houses, as well as industrial process controllers and a host of other applications.

The main practical use of fuzzy logic has been in the myriad of applications in Japan as process controllers. But the earliest fuzzy control developments took place in Europe.

FUZZY CONTROL SYSTEMS

The British engineer Ebrahim Mamdani was the first to use fuzzy sets in a practical control system, and it happened almost by accident. In the early 1970s, he was developing an automated control system for a steam engine using the expertise of a human operator. His original plan was to create a system based on Bayesian decision theory, a method of defining probabilities in uncertain situations that considers events after the fact to modify predictions about future outcomes.

The human operator adjusted the throttle and boiler heat as required to maintain the steam engine's speed and boiler pressure. Mamdani incorporated the operator's response into an intelligent algorithm (mathematical formula) that learned to control the engine. But as he soon discovered, the algorithm performed poorly compared to the human operator. A better method, he thought, might be to create an abstract description of machine behavior.

He could have continued to improve the learning controller. Instead, Mamdani and his colleagues decided to use an artificial intelligence method called a *rule-based expert system*, which combined human expertise with a series of logical rules for using the knowledge. While they were struggling to write traditional rules using the computer language Lisp, they came upon a new paper by Lotfi Zadeh on the use of fuzzy rules and algorithms for analysis and decision-making in complex systems. Mamdani immediately decided to try fuzziness, and within a "mere week" had read Zadeh's paper and produced a fuzzy controller. As Mamdani has written, "it was 'surprising' how easy it was to design a rule-based controller" based on a combination of linguistic and mathematical variables.

In the late 1970s, two Danish engineers, Lauritz Peter Holmblad and Jens-Jurgen Ostergaard, developed the first commercial fuzzy control system, for a cement kiln. They also created one for a lime kiln in Sweden, and several others.

Other Commercial Fuzzy Systems

The most spectacular fuzzy system functioning today is the subway in the Japanese city of Sendai. Since 1987, a fuzzy control system has kept the trains rolling swiftly along the route, braking and accelerating gently, gliding into stations, stopping precisely, without losing a second or jarring a passenger.

Japanese consumer product giants such as Matsushita and Nissan have also climbed aboard the fuzzy bandwagon. Matsushita's fuzzy vacuum cleaner and washing machine are found in many Japanese homes. The washing machine evaluates the load and adjusts itself to the amount of detergent needed, the water temperature, and the type of wash cycle. Tens of thousands of Matsushita's fuzzy camcorders are producing clear pictures by automatically recording the movements the lens is aimed at, not the shakiness of the hand holding it.

Sony's fuzzy TV set automatically adjusts contrast, brightness, sharpness, and color.

Nissan's fuzzy automatic transmission and fuzzy antilock brakes are in its cars.

Mitsubishi Heavy Industries designed a fuzzy control system for elevators, improving their efficiency at handling crowds all wanting to take the elevator at the same time. This system in particular captured the imagination of companies elsewhere in the world. In the United States, the Otis Elevator Company is developing its own fuzzy product for scheduling elevators for time-varying demand.

Since the Creator of Crispness, Aristotle, had a few doubts about its application to everything, it shouldn't be a surprise that other methods of dealing with instability also exist. Some of them are a couple of centuries old.

THE VALUE OF FUZZY SYSTEMS

Writing 20 years later, Ebrahim Mamdani noted that the surprise he felt about the success of the fuzzy controller was based on cultural biases in favor of conventional control theory. Most controllers use what is called the *proportional-integral-derivative* (PID) control law. This sophisticated mathematical law assumes linear or uniform behavior by the system to be controlled. Despite this simplification, PID controllers are popular because they maintain good performance by allowing only small errors, even when external disturbances occur threaten to make the system unstable.

In fact, PID controllers were held in such high repute that any alternative control method would be expected to be equally sophisticated (meaning complicated), what Mamdani calls the "cult of analyticity."

One of the "drawbacks" of fuzzy logic is that it works with just a few simple rules. In other words, it didn't fit people's expectations of what a "good" controller should be. And it certainly shouldn't be quick and easy to produce.

Despite the culture shock, fuzzy control systems caught on–faster in Japan than in the United States–because of two drawbacks of conventional controllers. First, many processes aren't linear, and they're just too complex to be modeled mathematically. Management, economic, and telecommunications systems are examples.

Second, even for the traditional industrial processes that use PID controllers, it's not easy to describe what the term *stability* means. As Mamdani has noted, the idea of requiring mathematical definition of stability has been an academic view that hasn't really been used in the workplace. There's no industry standard of "stability," and the various methods of describing it are recommendations, not requirements. In practical terms, the value of a controller is shown by prototype tests rather than stability analysis. In fact, Mamdani says, experience with fuzzy controllers has shown that they're often more robust and stable than PID controllers.

There are five types of systems where fuzziness is necessary or beneficial:

- Complex systems that are difficult or impossible to model
- Systems controlled by human experts
- Systems with complex and continuous inputs and outputs

- Systems that use human observation as inputs or as the basis for rules
- Systems that are naturally vague, such as those in the behavioral and social sciences

Advantages and Disadvantages

According to Datapro, the Japanese fuzzy logic industry is worth billions of dollars, and the total revenue worldwide is projected at about $650 million for 1993. By 1997, that figure is expected to rise to $6.1 billion. According to other sources, Japan currently is spending $500 million a year on Fuzzy Systems R&D. And it's beginning to catch on in the United States, where it all began.

Advantages of Fuzzy Logic for System Control

- Fewer values, rules, and decisions are required.
- More observed variables can be evaluated.
- Linguistic, not numerical, variables are used, making it similar to the way humans think.
- It relates output to input, without having to understand all the variables, permitting the design of a system that may be more accurate and stable than one with a conventional control system.
- Simplicity allows the solution of perviously unsolved problems.
- Rapid prototyping is possible because a system designer doesn't have to know everything about the system before starting work.
- They're cheaper to make than conventional systems because they're easier to design.
- They have increased robustness.
- They simplify knowledge acquisition and representation.
- A few rules encompass great complexity.

Its Drawbacks

- It's hard to develop a model from a fuzzy system.
- Though they're easier to design and faster to prototype than conventional control systems, fuzzy systems require more simulation and fine tuning before they're operational.
- Perhaps the biggest drawback is the cultural bias in the United States in favor of mathematically precise or crisp systems and linear models for control systems.

FUZZY DECISION-MAKING

Fuzzy decision-making is a specialized, language oriented fuzzy system used to make personal and business management decisions, such as purchasing cars and appliances. It's even been used to help resolve the ambiguities in spouse selection!

On a more practical level, fuzzy decision-makers have been used to optimize the purchase of cars and VCRs. The Fuji Bank has developed a fuzzy decision-support system for securities trading.

FUZZINESS AND ASIAN NATIONS

If the names Nissan, Matsushita, and Fuji Bank jumped out at you, there's a reason. As they indicate, Japan is the world's leading producer of fuzzy-based commercial applications. Japanese scientists and engineers were among the earliest supporters of Lotfi Zadeh's work and, by the late 1960s, had introduced fuzziness in that country. In addition, research on fuzzy concepts and products is enthusiastically pursued in China. According to one survey, there are more fuzzy-oriented scientists and engineers there than in any other country.

Why has fuzzy logic caught on so easily in Asian nations, while struggling for commercial success in the United States and elsewhere in the West? There are two possibilities.

One answer is found in the different traditional cultures. As you saw earlier, one of the hallmarks of Western culture is the Aristotelian either-or approach to thought and action. Individual competitiveness and a separation of human actions from the forces of nature have helped foster the early development of technology in Europe and the United States.

The culture of China and Japan developed with different priorities. Strength and success were accomplished through consensus and accommodation among groups. This traditional attitude, so perplexing to Americans, is basic to Japanese business transactions today, from the smallest firm to the largest high-tech company. In addition, the forces of nature were traditionally expected to be balanced between complementary extremes—the Yin-Yang of Zen is an example. Fuzzy logic is much more compatible with these tenets than with the mathematically oriented Western concepts.

Or it may be that the research-oriented government-industry establishment in Japan is simply more open to new ideas and approaches than in management- and bottom line-oriented Western firms.

FUZZY SYSTEMS AND UNCERTAINTY

Two broad categories of uncertainty methods are currently in use—probabilistic and nonprobabilistic. Probabilistic and statistical techniques are generally applied throughout the natural and social sciences and are used extensively in artificial intelligence. Several nonprobabilistic methods have been devised for problem solving, particularly "intelligent," computerized solutions to real-world problems. In addition to fuzzy logic, they include default logic, the Dempster-Shafer theory of evidence, endorsement-based systems, and qualitative reasoning.

E-MAIL FROM DR. FUZZY These other methods of dealing with uncertainty provide interesting context. But you don't have to understand them thoroughly to understand fuzziness.

Chapter 2

Fuzzy Numbers and Logic

Scene: a deli counter.

"I want a couple of pounds of sliced cheeses. Give me about a half-pound each of swiss, cheddar, smoked gouda, and provolone."

The clerk works at the machine for a while and comes back with four mounds. "I went a little overboard on the swiss. Is 9 oz. OK? There's 9 oz. of the cheddar too, and a tad under 8 oz. of the provolone. We only had about 7 oz. of the gouda. Is that close enough?"

"That's fine," the customer says.

Somewhere early in life, we all learned that

$$2 + 2 = 4$$

at least in school and cash transactions. With flash cards, Cuisenaire rods, or by rote, we also absorbed the messages that

$$2 - 2 = 0$$
$$2 \times 2 = 4$$
$$2 / 2 = 1$$

There's nothing wrong with these precise—or crisp—numerical values. But as the scene in the deli shows, they're not always necessary or appropriate. Sometimes fuzzy numbers are better. At the cheese counter, "about half a pound" turned out to be anywhere from 7 oz. to 9 oz. and the service was quicker than if the clerk had laboriously cut exactly 8 oz. of each type of cheese. With the gouda, in fact, exactly 8 oz. would have been impossible to produce. All in all, the customer ended up with "a couple of pounds," as planned.

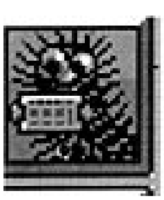
In this chapter, you'll delve more deeply into fuzziness, beginning with some basic concepts. The first of these is fuzzy numbers and fuzzy arithmetic operations. You'll also learn the fine art of creating fuzzy sets and performing fuzzy logical operations on them. And you'll discover how fuzzy sets, fuzzy rules of inference, and fuzzy operations differ from crisp ones. Finally, you'll begin learning the use of As-Do and As-Then problem-solving rules (the fuzzy equivalents of If-Then rules).

As always, Dr. Fuzzy will be available with more information and encouragement.

E-MAIL FROM DR. FUZZY

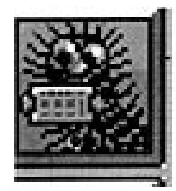
Why learn the inner workings of fuzzy sets and rules?

They're the power behind most fuzzy systems out here in the real world.



Throughout the chapter, you can make use of the doctor's own series of fuzzy calculators, contained on the disk that accompanies this book. Each calculator is fully operational. You can compute the examples in the book, use your own examples, or press the **e** button to automatically load random numbers. The Introduction to the book contains instructions for using the disk programs with Windows 3.1 or above. Portions of the text that are related to calculator operations are marked with Dr. Fuzzy's cartouche. The doctor also provides context-sensitive help on request from the calculator screen.

Figure 2.1: A crisp 8.



As they say in Dr. Fuzzy's family, you have to crawl before you can fly, so we're going to ease into the doctor's Fuzzy World Tour with some very elementary fuzzy arithmetic.

Fortunately, the doctor likes to make tracks on wheels. Open the first calculator, FuzNum Calc by clicking on the Trike icon, and let's get rolling.

FUZZY NUMBERS

Back at the deli, a crisp "half pound" (8 oz.) registers on the scale as shown in Figure 2.1. Deli's don't have fuzzy scales (the Dept. of Weights and Measures would frown). But if they did, "about a half pound" might register like the representation in Figure 2.2.

Now try your own hand at fuzzy arithmetic with FuzNum Calc.

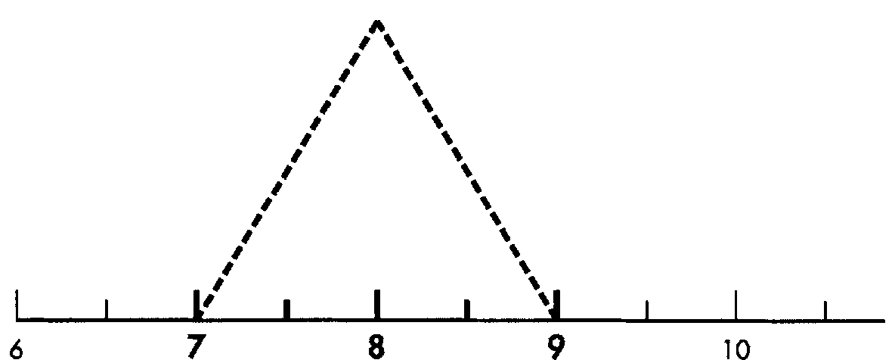


Figure 2.2: A fuzzy 8.

E-MAIL FROM DR. FUZZY

You can turn off any of the fuzzy calculators by clicking on its OFF button.

FUZZY SETS

As the fuzzy calculator showed, any fuzzy number can be represented by a triangle. If you think of the calculator's linear scale as the horizontal line (abscissa) of a graph, you can easily convert the diagram to the repesentation of a fuzzy set by adding a vertical scale (Figure 2.8):

The values in this set—7, 8, and 9—have various degrees of membership in the set of Eightness. For instance, 7 and 9 have the least degree of membership, while 8 has the greatest degree of membership. You might represent these degrees of membership as shown in Table 2.2.

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A triangular fuzzy set's apex has a membership value of 1. The base numbers have membership values of close to 0.

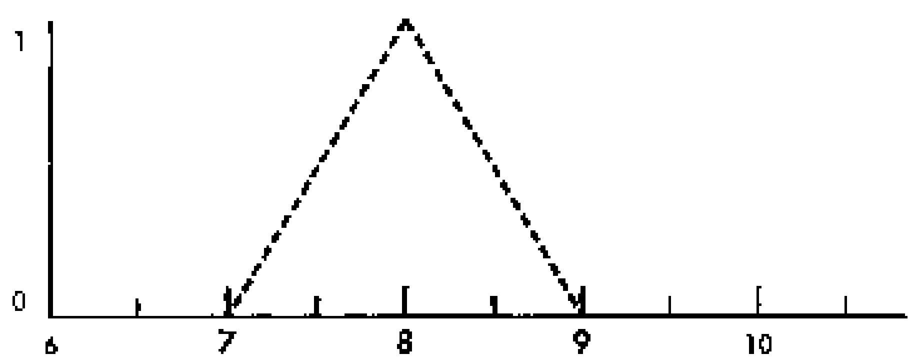


Figure 2.8: An example of fuzzy set of Eightness with a triangular membership function.

TABLE 2.2: The Set of "Eightness" with a Triangular Membership Function.

Member	Degree of Membership
7	.0
7.5	.5
8	1
8.5	.5
9	.0

The triangular membership function is the most frequently used function and the most practical, but other shapes are also used. One is the trapezoid, as shown in Figure 2.9. The trapezoid contains more information than the triangle.

A fuzzy set can also be represented by a quadratic equation (involving squares, n^2, or numbers to the second power), which produces a continuous curve. Three shapes are possible, named for their appearance—the S function, the pi function, and the Z function (Figure 2.10).

Like other types of sets, fuzzy sets can be made to interact with each other to produce a usable result.

Most people have been exposed to classical set theory in school. In the world of fuzziness, classical set theory is called *crisp set theory,* in which set membership is limited to 0 or 1.

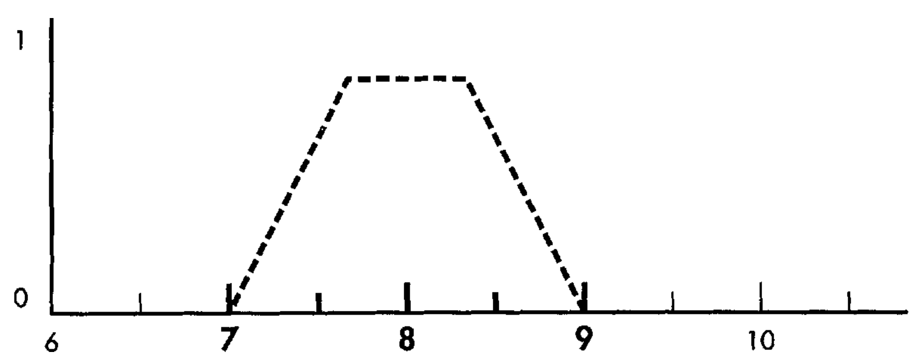


Figure 2.9: A fuzzy set of Eightness with a trapezoidal membershlp function.

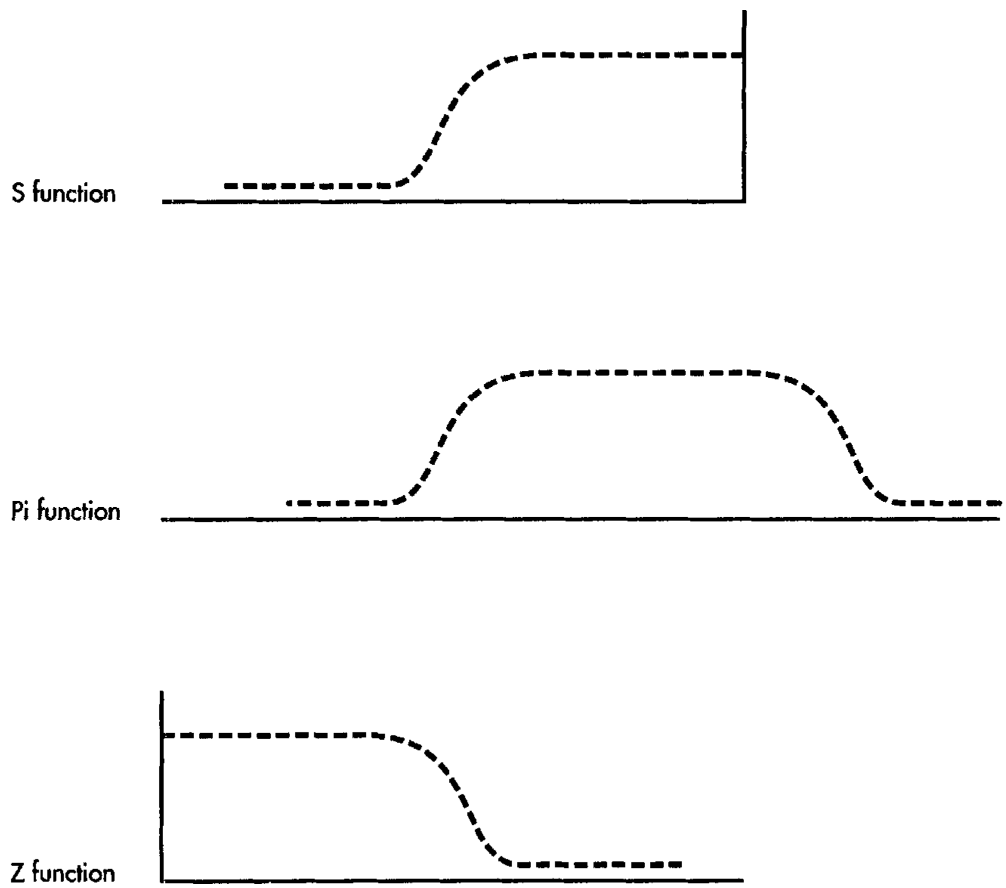


Figure 2.10: Graphs of the S function, the pi function, and the Z function.

Set Theory

The basic purpose of a set is to single out its elements from those in its domain or "universe of discourse" (Figure 2. 11a). The relationship between two sets has two possibilities. Either they're partners merged in a larger entity or the relationship consists of the elements that they have in common.

Sets as partners (see Figure 2.11b) is called a *disjunction* (for single-element, or atomic, sets), using the symbol $\vee$, or a union (for multielement sets), using the symbol $\cup$. The disjunction or union of two sets means that any element belonging to either of the sets is included in the partnership. In the fuzzy world, this partnership expresses the maximum value for the two fuzzy sets involved.

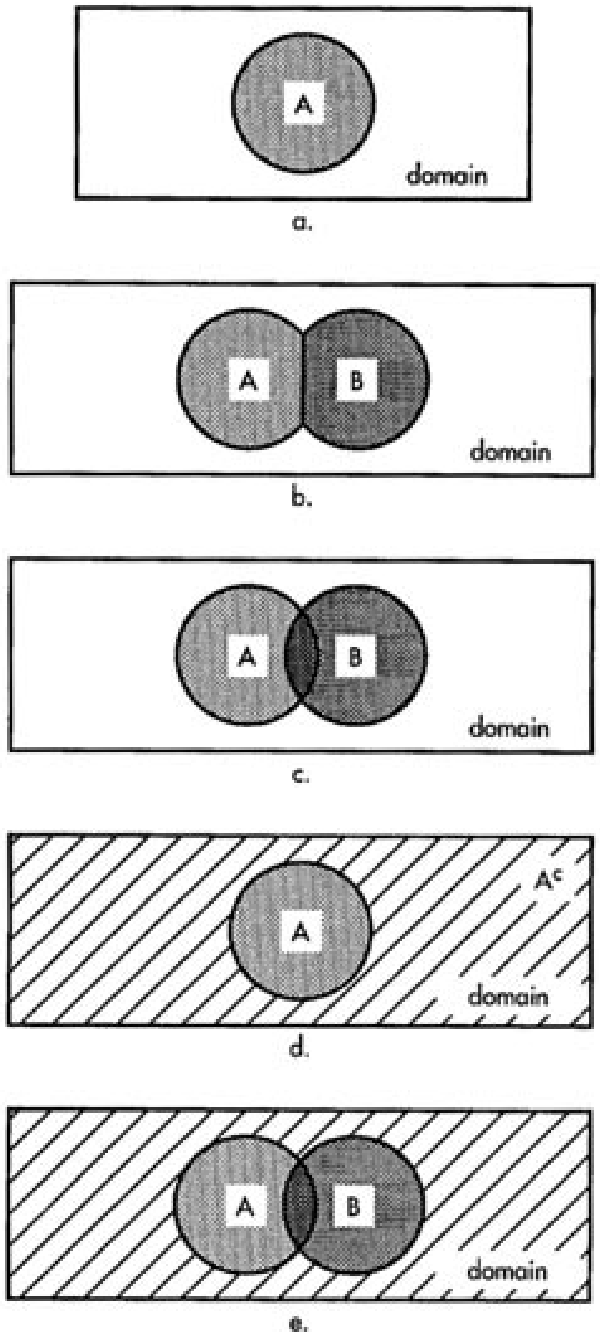


Figure 2.11: Crisp set operations: (a) Set A in a domain, (b) disjunctionor union of Set A and Set B, (c) conjunction or intersection of Set A and Set B, (d) complement of Set A and Set Not-A in its domain, and (e) difference of Set A and Set B.

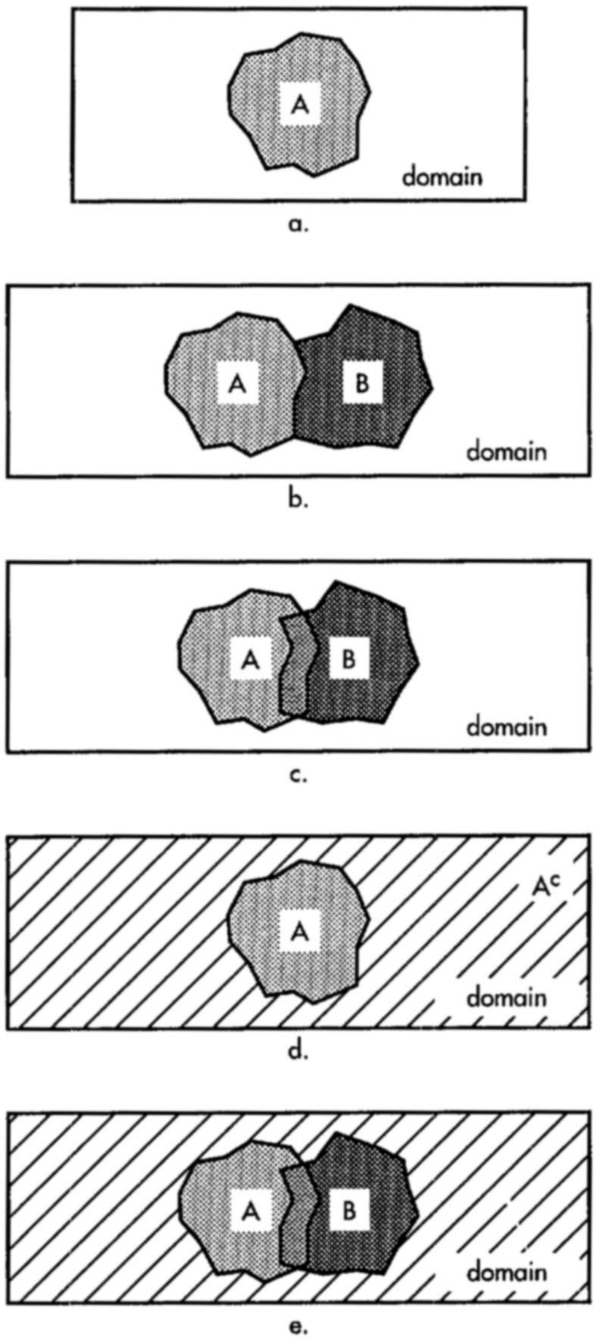


Figure 2.12: Fuzzy Set Operations: (a) fuzzy Set A in a domain, (b) disjunction or union (MAX) of fuzzy Set A and fuzzy Set B, (c) conjunction or intersection (MIN) of fuzzy Set A and fuzzy Set B, (d) complement of fuzzy Set A and Set Not-A in its domain, and (e) difference of fuzzy Set A and fuzzy Set B.

Set elements in common (Figure 2.11c) is called a conjunction (for single-element sets) or intersection $\cap$ (for multielement sets). A conjunction or intersection makes use of only those aspects of Set A and Set B that appear in both sets. In the fuzzy world, this partnership expresses the minimum value for the two fuzzy sets involved.

The part of the domain not in a set can also be characterized (Figure 2.11d)—what's called not-A (A^C). Not-A can also be written ~A or ≠A.

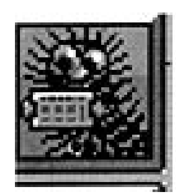
Set theory is closely linked to an operation in logic—the use of mathematics to find truth or correctness—called *implication*. (There's more on logical operations later in the chapter.) Implication is a statement that if the first of two expressions is true, then the second one is true also. For example, given the expressions A and B, if A is true, then B is also true. In other words,

A implies B

This can also be written

$$A \rightarrow B$$

As you've already experienced, fuzziness provides a great variety of ways for sets to interact—much more so than crispness. Looked at in this way, fuzzy sets are the more general way of approaching sets, and crisp sets are a special case of that generality. Figure 2.12 represents fuzzy versions of the principal set operations.



Set theory, fuzzy and crisp, can be better understood through use of another of the fuzzy calculators, the one named UniCalc. It calculates operations on single element sets. Change vehicles—or calculators—by clicking on the Bicycle icon to open UniCalc.

Touring UniCalc

UniCalc (Figure 2.13) provides a numeric / decimal keypad, the set operators conjunction ($\wedge$), disjunction ($\vee$), not–A (~A), not–B (~B), and implication (the arrow key). To enter single-element values for Set A, click on the box by A and then on the desired keypad numbers. Follow the same procedure for Set B. You can enter any value between 0 and 1.