Preliminaries

The reduction technique revisited: a way to show that a problem is \( \text{NP} \)-complete is to use the deduction rule below.

\[
\begin{align*}
L & \text{ is } \text{NP}-\text{complete} \\
L' & \text{ is in } \text{NP} \\
\text{There is a mapping } f \text{ such that } x \in L \iff f(x) \in L' & \text{ holds for all } x \\
f(x) & \text{ can be computed in logarithmic space w.r.t. } |x| \\
L' & \text{ is } \text{NP}-\text{complete}
\end{align*}
\]

A Proof that FEEDBACK VERTEX SET is \( \text{NP} \)-complete

Preliminaries

The problem NODE COVER is the following:

Given an undirected graph \( G = \langle V, E \rangle \) and an integer \( B \leq |V| \), is there a subset \( V' \subseteq V \), \( |V'| \leq B \), such that each edge in \( G \) has at least one of its endpoints in \( V' \).

We know that NODE COVER is \( \text{NP} \)-complete (see page 190 in Papadimitriou’s book).

The Problem and a Solution

Show that the following problem, called FEEDBACK VERTEX SET, is \( \text{NP} \)-complete:

Given a directed graph \( G = \langle V, E \rangle \) and an integer \( B \leq |V| \), is there a subset \( V' \subseteq V \) such that (i) \( |V'| \leq B \), and (ii) every directed circuit in \( G \) includes at least one vertex from \( V' \)?

Note that an equivalent definition of the problem is: Given a directed graph \( G = \langle V, E \rangle \) and an integer \( B \leq |V| \), is there a subset \( V' \subseteq V \) such that (i)
\(|V'| \leq B\), and (ii) the directed graph \(G' = (V \setminus V', E \cap (V \setminus V') \times (V \setminus V'))\) has no cycles?

Obviously FEEDBACK VERTEX SET is in \(\text{NP}\): (i) guess the set \(V'\) such that \(|V'| \leq B\), and (ii) use, e.g., Tarjan’s algorithm for finding strongly connected components of a graph to check whether \(G' = (V \setminus V', E \cap (V \setminus V') \times (V \setminus V'))\) has any cycles.

We show the \(\text{NP}\)-hardness by reducing from the \(\text{NP}\)-complete problem NODE COVER. Given an input \(G; B\) for NODE COVER, where \(G = (V, E)\) is an undirected graph and \(B \leq |V|\) is an integer, the corresponding input for FEEDBACK VERTEX SET is \(\hat{G}; B\), where \(\hat{G}\) is the directed graph \(\hat{G} = (V, \hat{E} = \{\{v_i, v_j\} \mid \{v_i, v_j\} \in E\})\). That is, \(\hat{G}\) is \(G\) interpreted as a directed graph.

Now if \(G\) has a node cover \(V'\), \(|V'| \leq B\), then the directed graph \(G' = (V \setminus V', \hat{E} \cap (V \setminus V') \times (V \setminus V'))\) has no edges and thus cannot have any cycles. Thus if \(V'\) is a node cover for \(G\), then \(V'\) is a feedback vertex set for \(\hat{G}\).

On the other hand, assume that \(\hat{G}\) has a feedback vertex set \(V'\), \(|V'| \leq B\). Then for each edge pair \(\langle v_i, v_j \rangle\) and \(\langle v_j, v_i \rangle\) between the nodes \(v_i\) and \(v_j\), at least one of \(v_i, v_j\) has to be in \(V'\) (otherwise \(G' = (V \setminus V', \hat{E} \cap (V \setminus V') \times (V \setminus V'))\) would have a cycle \(v_i \rightarrow v_j \rightarrow v_i\)). Therefore \(V'\) is also a node cover for \(G\).

To sum up, \(V'\) is a node cover of \(G\) iff \(V'\) is a feedback vertex set of \(\hat{G}\). Obviously the reduction can be computed in logarithmic space.

**A Proof that PARTITION INTO TRIANGLES is NP-complete**

**Preliminaries**

Graphs in this exercise are assumed to be undirected and not containing self-loops.

The problem EXACT COVER BY 3-SETS is the following:

We are given a family \(F = \{S_1, \ldots, S_n\}\) of subsets of a set \(U\), such that \(|U| = 3m\) for some integer \(m\), and \(|S_i| = 3\) for all \(i\). The question is whether there are \(m\) sets in \(F\) that are disjoint and have \(U\) as their union.
We know that EXACT COVER BY 3-SETS is \( \mathbf{NP} \)-complete (see page 201 in Papadimitriou’s book).

The Problem and a Solution

Show that the following problem, called PARTITION INTO TRIANGLES, is \( \mathbf{NP} \)-complete:

Given a graph \( G = (V, E) \) with \( |V| = 3q \) for an integer \( q \), is there a partition of \( V \) into \( q \) mutually disjoint sets \( V_1, V_2, \ldots, V_q \) of three vertices each such that, for each \( V_i = \{v_{i,1}, v_{i,2}, v_{i,3}\} \), the three edges \( \{v_{i,1}, v_{i,2}\}, \{v_{i,1}, v_{i,3}\}, \text{ and } \{v_{i,2}, v_{i,3}\} \) all belong to \( E \)?

Clearly PARTITION INTO TRIANGLES is in \( \mathbf{NP} \) because we can first nondeterministically guess a partition of \( V \) and then check (in deterministic polynomial time) that each \( V_i \) fulfills the triangle condition.

We show the \( \mathbf{NP} \)-hardness by reducing from the \( \mathbf{NP} \)-complete problem EXACT COVER BY 3-SETS. Assume that we are given a set \( U \) such that \( |U| = 3m \) for some integer \( m \) and a family \( F = \{S_1, \ldots, S_n\} \) of subsets of \( U \) such that \( |S_i| = 3 \) for all \( i \). We now construct a graph \( G = (V, E) \) with \( |V| = 3q' \) such that \( G \) can be partitioned into \( q' \) triangles iff \( F \) contains an exact cover for \( U \).

First, it can be safely assumed that each element \( u \in U \) appears in at least one set \( S_i \) in \( F \): if this is not the case, then \( F \) cannot have an exact cover for \( U \) and we can simply output a simple constant graph that cannot be partitioned into triangles. Otherwise, we substitute for each set \( S_i = \{u, v, w\} \) appearing in \( F \) the collection \( E_i \) of 18 edges shown below.

![Diagram](image-url)
The graph \( G = \langle V, E \rangle \) is now defined by

\[
V = U \cup \bigcup_{i=1}^{n} \{a_{i,j} \mid 1 \leq j \leq 9\}
\]

\[
E = \bigcup_{i=1}^{n} E_i.
\]

Now \( |V| = |U| + 9n = 3m + 9n \) and thus \( q' = m + 3n \). Note that only the \( U \)-vertices may be shared by different \( E_i \)s.

Assume that \( F' \subseteq F \) is an exact partition of \( U \) (and thus \( |F'| = m \)). Then, if \( S_i = \{u, v, w\} \in F' \), take \( \{a_{i,3}, a_{i,6}, a_{i,9}\} \), \( \{v, a_{i,4}, a_{i,5}\} \) and \( \{w, a_{i,7}, a_{i,8}\} \) to belong to the triangle partition of \( G \) and, if \( S_i = \{u, v, w\} \notin F' \), choose \( \{a_{i,1}, a_{i,2}, a_{i,3}\} \), \( \{a_{i,4}, a_{i,5}, a_{i,6}\} \) and \( \{a_{i,7}, a_{i,8}, a_{i,9}\} \) to belong to the triangle partition of \( G \). Clearly the result is a valid triangle partition of \( G \) because the partition is disjoint, each vertex is in the partition and the partition has \( q' = m + 3n \) sets.

To show the opposite direction, assume that \( V \) can be partitioned into \( q' = m + 3n \) triangles \( V_1, V_2, \ldots, V_{q'} \). Then the corresponding exact cover of \( U \) is given by choosing those \( S_i \) of \( F \) such that \( \{a_{i,3}, a_{i,6}, a_{i,9}\} = V_k \) for some \( 1 \leq k \leq q' \). This is justified as follows. For each \( u \in U \) there is exactly one triangle into which \( u \) belongs, let's say it was \( \{u, a_{i,1}, a_{i,2}\} \). Then \( \{a_{i,3}, a_{i,6}, a_{i,9}\} \), \( \{v, a_{i,4}, a_{i,5}\} \) and \( \{w, a_{i,7}, a_{i,8}\} \) for \( S_i = \{u, v, w\} \) must also be in the triangle partition. Furthermore, since no other \( \{u, a_{j,x}, a_{j,y}\} \) can be in the triangle partition, the corresponding \( \{a_{j,3}, a_{j,6}, a_{j,9}\} \) cannot be in the triangle partition. Therefore each \( u \) appears in exactly one chosen \( S_i \). This also implies that we choose exactly \( m \) \( S_i \)s.

We have thus showed that \( G \) can be partitioned into \( q' = m + 3n \) triangles iff \( F \) contains an exact cover of \( m \) sets for \( U \).

To show that the reduction can be carried out in logarithmic space, note that the reduction is made by local substitutions. We therefore need \( 3 \) registers of length \( \log(3m) \) to remember the elements in the currently processed set \( S_i \) of \( F \) and one counter of length \( \log n \) to remember the number \( i \) of the set. We then just output the edges in \( E_i \) to the output tape.