Comparing Hasse Diagrams

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Abstract— **In this paper, we investigate the applicability of some relational strength measures when comparing and analyzing Hasse diagrams. The structures used were derived from a particular line of research at our Department, ongoing as of this writing, that deals with human perception of an urban environment.**

I. Introduction

A methodology for comparing Hasse diagrams is presented in this paper. The Hasse diagrams used are derived from data collected in an Urbanistics project, described in [1], that investigates human perception of an urban environment. Knowledge was elicited by allowing interviewees to "relate" bipolar constructs, containing both positive and negative poles, with tangible landmarks in the vicinity of their neighborhood. The triangular products of these *fuzzified* relations were used to derive, say, construct–construct relations from which Hasse diagrams are generated. These diagrams represent certain knowledge structures that the interviewee possesses.

II. Hasse Diagrams

Relations are analysed in order to extract deep structure from them. One way to do this is to derive certain closures and interiors depending on the relational properties under investigation. Perhaps one of the most interesting group of properties is that exhibited by *orders* and *preorders*. Normally, Hasse diagrams are used to graphically illustrate the different categories hidden by such relations.

A. Processing Methodology

The data used in this investigation were processed using a conceptual procedure presented in [2].Given some fuzzy relation $T \subseteq X \times Y$, fuzzy local preorders (say, $R \subseteq X \times X$) were derived by taking the local preorder closure of the triangle product[1] $(T \triangleleft T^{-1})$.

The following *Hassefication procedure*, given in [2], is then used

1. Take an α–cut R_α.
2. Form $S = \text{sym int} R_\alpha$. This is a *local equivalence*.
3. Remove the zero–class C_0, consisting of all x's unrelated by S to any elements.
4. Let E be the *factor set* of $X \setminus C_0$ according to S.
5. Denote by $\preceq$ the *factor relation* R_α/S, an *order*.
6. Let $P = (E, \preceq)$. Draw the Hasse diagram $H(P)$.

B. Graph Theoretic Concepts

A pair $P = (E, \preceq)$ is a (partially) ordered set if for all $x, y, z \in E$

1. $x \preceq x$
2. $x \preceq y$, $y \preceq x$ implies $x = y$
3. $x \preceq y$, $y \preceq z$ implies $x \preceq z$

$P = (E, \preceq)$ can also be interpreted as a *directed graph* $D = D(P)$ defined on a set E of vertices and with edges of the form (x, y) whenever $x \preceq y$. The *comparability graph* $G = G(P)$ is the undirected version of $D(P)$. Hence, $G(P)$ has vertex set E and edges $\{x, y\}$ whenever $x \preceq y$. Note that in cases when the relation is a *strict order*, $D(P)$ and $G(P)$ may be defined without loops [3].

We say that y covers x if $x \prec y$ and there is no $z \in E$ such that $x \prec z \prec y$. Hence, the complete information about P is contained in the Hasse diagram $H = H(P)$, namely the subgraph of $D(P)$ retaining just the covering edges. Usually $H(P)$ is drawn as an undirected graph with the understanding that the orientation is from "bottom" to "top".

C. Choosing Alpha–Cuts

By taking various alpha–cuts on the relations considered, one can note the relationship between the cardinality of the factor set $|X|$, the mean *fuzzy cardinality*[2] of

[1] This study was restricted to the use of the Kleene–Dienes fuzzy implication operator.

[2] We denote a formulation for the mean fuzzy cardinality of a fuzzy relation R by

$$\bar{R} = \frac{\Sigma\text{Count}(R)}{|X|^2}$$

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TABLE I
NUMBER OF STRATA AND CARDINALITIES FOR VARIOUS ALPHA–CUTS CONSIDERED

Original Relation	*α–cut Value*	*No. of Strata*	*Factor Set* $\|X\|$	*Factor Relation* $\hat{R}$
gr17	0.40	2	2	0.750
	0.50	3	4	0.562
	0.60	5	12	0.312
	0.80	3	10	0.280
	1.00	2	6	0.277
gr18	0.20	3	4	0.542
	0.40	2	4	0.375
	0.60	2	6	0.277
	0.80	2	10	0.180
	1.00	3	14	0.183
gr21	0.40	1	1	1.000
	0.50	4	6	0.472
	0.60	4	7	0.428
	0.80	2	3	0.555

the factor relation, and the number of strata in the corresponding Hasse diagram. Notice from Table I that the cardinality of the factor set, which is the number of nodes in the Hasse diagram, peaks at about the same α–cut that the number of strata peaks. We based our choice of values for α–cuts on this observation. The α values considered are:[3]

- $\alpha = 0.60$ and $\alpha = 0.80$ for gr17
- $\alpha = 0.60$ and $\alpha = 0.80$ for gr18
- $\alpha = 0.50$ and $\alpha = 0.60$ for gr21

The corresponding Hasse diagrams for the α–values chosen are illustrated in Figures 1 to 6. To identify the Hasse diagrams considered, "$H_{i,j}$" will denote the Hasse diagram for gr*i* at an alpha–cut of $\alpha = j$. So, $H_{18,0.60}$ denotes a Hasse diagram for gr18 at $\alpha = 0.60$.

III. A METHOD FOR MAPPING HASSE DIAGRAMS

In [6], fuzzy relational products were used in comparing and verifying the correctness of medical knowledge structures. Several alpha–cuts and fuzzy implication operators were considered in that study.

Hasse diagrams in this study represent deep structure: a relation between (equivalence) classes. Let $P_A = (A, \preceq_A)$ and $P_B = (B, \preceq_B)$ be two (partially) ordered sets obtained through the Hassefication procedure in the previous section. Then, A and B are subsets of 2^X for some set X; i.e. $A, B \subseteq 2^X$. We can define a homomorphism between the Hasse diagrams $H_A = H(P_A)$ and $H_B = H(P_B)$ to

[3]The grs's indicate data from the [illegible] project representing knowledge structures elicited from interviewees. Any other detailed specification is irrelevant to the present study.

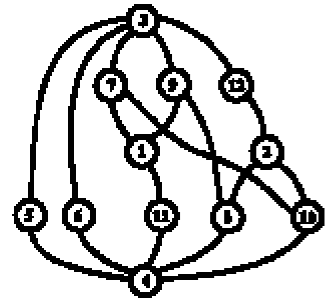

Figure 1: $H_{17,0.60}$, Hasse diagram for gr17 at $\alpha = 0.60$

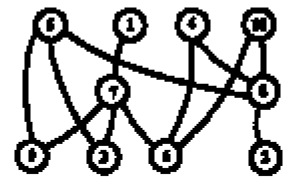

Figure 2: $H_{17,0.80}$, Hasse diagram for gr17 at $\alpha = 0.80$

be a function $\mathcal{F} : H_A \to H_B$ that maps from H_A to H_B consisting of a pair $\mathcal{F} = \langle f, g \rangle$ where

- $f : A \to B$ is a mapping between the factor sets of H_A and H_B ($f \subseteq 2^X \times 2^X$); and
- g is a partial mapping from paths in H_A into paths in H_B (based on the *factor relations* $\preceq_A$ and $\preceq_B$).

Defining the mappings f and g are discussed in the next two sections. The notations are a modification of that used in [4].

A. Mapping Nodes Between Hasse Diagrams

There are many ways to define a mapping f between the factor sets of two Hasse diagrams under consideration. For example, one may use the notion of containment. In our case, the mapping f was based on the notion of overlap.

We wish to form $\mathcal{F} : H_A \to H_B$ to be some mapping between the Hasse diagrams H_A and H_B. Define f as follows: match a node $\bar{x} \in A$ with a node $\bar{y} \in B$, which are of course (equivalence) classes[4], if the two nodes overlap.

For simplicity, we can use the following function to describe f:

$$f(\bar{x}, \bar{y}) = \begin{cases} 1.00, & \text{if } \bar{x} \cap \bar{y} \neq \emptyset; \text{ and} \\ 0.00, & \text{otherwise} \end{cases} \quad (1)$$

From Figure 7, since $\bar{x} \cap \bar{y} = \{d\} \neq \emptyset$, then $f(\bar{x}, \bar{y}) = 1.00$. Similarly, $\bar{w} \cap \bar{z} = \{a\} \neq \emptyset$, and so $f(\bar{w}, \bar{z}) = 1.00$. Of course, $f(\bar{x}, \bar{z}) = f(\bar{w}, \bar{y}) = 0.00$ since both these pairs of nodes do not overlap.

[4]The use of the overhead bar was to emphasize that $\bar{x}$ and $\bar{y}$ are actually sets; $\bar{x}, \bar{y} \in 2^X$.

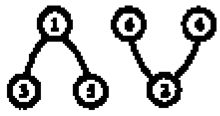

Figure 3: $H_{18,0.60}$, Hasse diagram for gr18 at $\alpha = 0.60$

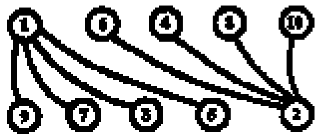

Figure 4: $H_{18,0.80}$, Hasse diagram for gr18 at $\alpha = 0.80$

A more restrictive mapping may be derived by fuzzifying the above function further. For example, we can use the notion of cardinalities to define f as

$$f(\bar{x}, \bar{y}) = \frac{|\bar{x} \cap \bar{y}|}{|\bar{x} \cup \bar{y}|} \qquad (2)$$

In this case then

$$f(\bar{x}, \bar{y}) = \frac{|\{d\} \cap \{d\}|}{|\{d\}|} = 1.00 \quad \text{and}$$
$$f(\bar{w}, \bar{z}) = \frac{|\{a\} \cap \{a,b,c,e,f\}|}{|\{a,b,c,e,f\}|} = 0.20.$$

Equation (1) was used in defining f for the analyses presented in this paper.

B. Mapping Paths Between Hasse Diagrams

Similarly, there are a number of ways to define the mapping g. This mapping will depend on the node mapping f. Consider all $\bar{x}, \bar{w} \in A$ such that $f(\bar{x}) = \bar{y}$, $f(\bar{w}) = \bar{z}$ for $\bar{y}, \bar{z} \in B$. If $(\bar{x}, \bar{w}) \in P_A$ and $(\bar{y}, \bar{z}) \in P_B$ then we can match these two paths. An implication operator may be used in order to derive the degree to which these paths are mapped:

$$g\left((\bar{x}, \bar{w}), (\bar{y}, \bar{z})\right) = \deg\left(\bar{x} P_A \bar{w} \rightarrow \bar{y} P_B \bar{z}\right) \qquad (3)$$

From Figure 7, P_A and P_B are crisp relations. This results from the Hassefication procedure used in the Urbanistics project: the factor sets and factor relations are crisp.

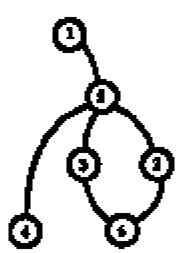

Figure 5: $H_{21,0.50}$, Hasse diagram for gr21 at $\alpha = 0.50$

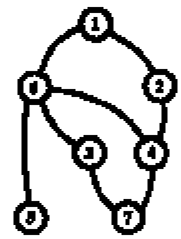

Figure 6: $H_{21,0.60}$, Hasse diagram for gr21 at $\alpha = 0.60$

Hence, $g\left((\bar{x}, \bar{w}), (\bar{y}, \bar{z})\right) = \deg\left(\bar{x} P_A \bar{w} \rightarrow \bar{y} P_B \bar{z}\right) = 1.00$. In this case, the choice of fuzzy implication operator to use becomes irrelevant.

A fuzzier value for g may be computed if instead of using the factor relations, say $\bar{x} P_A \bar{w}$, we use the local preorder closure R_A to derive $\bigvee x R_A w$ for all $x \in \bar{x}$ and $w \in \bar{w}$. Referring to Figures 8 and 9

$$\bar{x} P_A \bar{w} = \bigvee_{x \in \bar{x}, w \in \bar{w}} x R_A w = d R_A a = 0.60;$$
$$\bar{y} P_B \bar{z} = \bigvee_{y \in \bar{y}, z \in \bar{z}} y R_B z$$
$$= \bigvee \{d R_A a, d R_A b, d R_A c, d R_A e, d R_A f\}$$
$$= \bigvee \{1.0, 0.8, 0.6, 0.6, 1.0\} = 1.00$$

And so

$$g\left((\bar{x}, \bar{w}), (\bar{y}, \bar{z})\right) = \deg\left(0.60 \rightarrow 1.00\right)$$
$$= \left((1 - 0.60) \vee 1.00\right) = 1.00.$$

Equation (3) was used for the mappings derived in this study. Although there are other ways to make the path mappings more restrictive, the comparison and investigation of these and other approaches are beyond the scope of the current investigation.

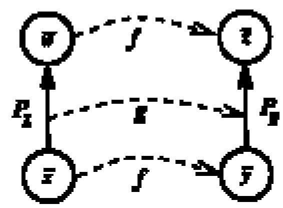

Figure 7: Sample portion of two Hasse diagrams

NOTE: This figure was derived from the Hasse diagram mapping $\mathcal{F}: H_{17,0.50} \rightarrow H_{18,0.60}$ (refer to Figures 2 and 3) where $\bar{x} = \{d\}$, $\bar{w} = \{a\}$, $\bar{y} = \{d\}$, and $\bar{z} = \{a, b, c, e, f\}$. Nodes $\bar{x}$ and $\bar{w}$ are actually nodes 3 and 5, respectively, in $H_{17,0.50}$. While $\bar{y}$ and $\bar{z}$ are nodes 3 and 1, respectively, in $H_{18,0.60}$.

	b		c		d				a		e		f	
b	.6	0	.6	0	.6	0	.5	0	.6	0	.6	0	.4	0
	.5	.6	.6	.5	.5	.6	.5	.5	.5	.5	.5	.5	.5	.6
c	.5	0	1	0	.5	0	.5	0	.5	0	.5	0	.4	0
	.6	.6	.6	1	.6	.8	.8	.6	1	.6	.6	.8	.6	.8
d	.6	0	.8	0	.8	0	.5	0	.8	0	.8	0	.4	0
	.5	.6	.6	.5	.5	.8	.5	.5	.5	.5	.5	.5	.5	.6
	.5	0	.6	0	.5	0	.6	0	.5	0	.5	0	.4	0
	.5	.5	.4	.5	.5	.5	.5	.6	.5	.5	.5	.5	.5	.5
a	.5	0	.6	0	.5	0	.5	0	1	0	.5	0	.4	0
	.5	.6	1	.6	.5	.6	.5	.5	.5	1	1	.5	.5	.8
e	.5	0	.8	0	.5	0	.5	0	.5	0	1	0	.4	0
	.5	.6	.6	.5	.5	.8	.5	.5	1	.5	.5	1	.5	.8
f	.6	0	.8	0	.6	0	.5	0	.8	0	.8	0	.6	0
	.5	.4	.6	.4	.5	.4	.5	.4	.5	.4	.5	.4	.4	.8

Figure 8: Local preorder closure of gr17.

	b		c		d				a		e		f	
b	1	0	.6	0	0	0	.4	0	1	0	.6	0	1	0
	.2	1	.2	.8	.2	.8	.2	.8	.2	.8	.2	.8	.2	.8
c	.8	0	1	0	0	0	.4	0	1	0	.6	0	1	0
	.2	.6	.2	1	.2	.6	.2	.6	.2	.6	.2	.6	.2	.6
d	.8	0	.8	0	1	0	.4	0	1	0	.6	0	1	0
	.2	0	.2	0	0	1	.2	0	.2	0	.2	0	.2	0
	.8	0	.6	0	0	0	1	0	1	0	.8	0	1	0
	.2	.4	.2	.4	.2	.4	.2	1	.2	.4	.2	.4	.2	.4
a	.8	0	.6	0	0	0	.4	0	1	0	.6	0	.8	0
	.2	1	.2	1	.2	1	.2	1	.2	1	.2	1	.2	1
e	.8	0	.6	0	0	0	.4	0	1	0	1	0	1	0
	.2	.6	.2	.6	.2	.6	.2	.6	.2	.6	.2	1	.2	.6
f	.8	0	.6	0	0	0	.4	0	1	0	.6	0	1	0
	.2	1	.2	1	.2	1	.2	1	.2	.8	.2	1	.2	1

Figure 9: Local preorder closure of gr18.

IV. Summary of Results

Mappings were derived both *within Hasse diagrams* (the diagrams considered come from the same data source, e.g. both from gr17) and *between Hasse diagrams* (coming from different sources). In Table II, the cardinalities of both the original factor sets and factor relations are summarized, along with the values for the corresponding *induced directed subgraphs* for each mapping considered in this paper. Recall that the factor sets are denoted by A and B and the resultant factor relations are P_A and P_B. We can deduce the following about the mappings used if we focus on the values for the induced subgraphs rather than the original graphs:

1. $|f| \geq \max(|induced\ A|, |induced\ B|)$; and
2. $|g| \geq \max(|induced\ P_A|, |induced\ P_B|)$.

This pattern is observed when considering the induced subgraphs. Whereas no observable patterns are observed when relying on the original graphs.

In order to derive an estimate of the "goodness of fit" of the mappings presented in this paper, the following

TABLE II
Scalar cardinalities for original and induced graphs

Mapping $\mathcal{F}: H_A \rightarrow H_B$	H_A original	H_A induced	H_B original	H_B induced
$H_{17,0.80} \rightarrow H_{17,0.60}$	12,45	10,28	10,26	10,28
$H_{18,0.80} \rightarrow H_{18,0.40}$	6,10	6,10	10,18	10,18
$H_{21,0.50} \rightarrow H_{21,0.60}$	6,17	6,17	7,21	7,21
$H_{17,0.80} \rightarrow H_{18,0.60}$	12,45	12,32	6,10	6,10
$H_{17,0.80} \rightarrow H_{18,0.40}$	10,28	10,24	6,10	4,6
$H_{18,0.40} \rightarrow H_{21,0.50}$	6,10	3,5	6,17	6,16
$H_{18,0.40} \rightarrow H_{17,0.80}$	10,18	10,12	12,45	12,18
$H_{21,0.50} \rightarrow H_{17,0.80}$	6,17	6,12	12,45	6,11
$H_{21,0.60} \rightarrow H_{18,0.80}$	7,21	7,19	6,10	3,5

NOTE: The numbers n, m indicate cardinalities for the factor set and for the factor relation, respectively.

methodology was used: obtain a measure for both the node maps and the path maps for each graph pair by dividing the scalar cardinality of the induced graph by the scalar cardinality of the original graph. Hence,

- for the node mapping, f, use

$$\max\left(\frac{|induced\ A|}{|original\ A|}, \frac{|induced\ B|}{|original\ B|}\right)$$

- for the path mapping, g, use

$$\max\left(\frac{|induced\ P_A|}{|original\ P_A|}, \frac{|induced\ P_B|}{|original\ P_B|}\right)$$

These values are listed in Table III. Notice that the values obtained for the mappings *within Hasse diagrams* indicate excellent fit. This supports the plausibility of the mapping method introduced in this paper. The table also seems to indicate that of all the mappings *between Hasse diagrams* considered, the one with the best fit is $H_{17,0.80} \rightarrow H_{18,0.60}$.

TABLE III
Estimates of "goodness of fit" for Hasse mappings

Mapping $\mathcal{F}: H_{i,j} \rightarrow H_{k,l}$	Nodes Map f	Links Map g
$H_{17,0.60} \rightarrow H_{17,0.60}$	1.000	1.000
$H_{18,0.60} \rightarrow H_{18,0.60}$	1.000	1.000
$H_{21,0.50} \rightarrow H_{21,0.60}$	1.000	1.000
$H_{17,0.80} \rightarrow H_{18,0.60}$	1.000	1.000
$H_{17,0.80} \rightarrow H_{18,0.90}$	1.000	0.857
$H_{18,0.60} \rightarrow H_{21,0.80}$	1.000	0.941
$H_{18,0.40} \rightarrow H_{17,0.60}$	1.000	0.666
$H_{21,0.50} \rightarrow H_{17,0.40}$	1.000	0.705
$H_{21,0.60} \rightarrow H_{18,0.90}$	1.000	0.004

V. Conclusions and Recommendations

The kind of mappings presented in this paper may be used to compute certain *congruences* between structures. In [6] and [7] the significance of the congruence between the *aspatial* and *spatial* structures that constitute a hyper-system for urban knowledge representation is discussed. Congruences between *actual* and *normative* structures are also mentioned. In particular, the *Structure Comparator* of the proposed General Meta-Knowledge Base in [7] may use these formulations to "assess the overall degree of congruence between two given structures".

Congruence may be used to determine which groups of people share the same view of their urban surroundings. This may also be used to approximate degrees of satisfaction with the urban environment. Urban planners could use such information to determine which part of the city needs more immediate attention. It is hoped that the methodology presented in this paper will eventually be embodied in a working system that will automate the analysis and diagnosis of data from certain urban studies.

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